Proper cones are manifolds with boundary

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Abstract

We introduce the relevant concepts and show that any proper cone in an n-dimensional real Hilbert space is an n-dimensional topological manifold whose topological boundary is itself a topological manifold of dimension $n-1$.

1 Reference

Throughout these notes, "Lee" refers to *Introduction to Topological Manifolds* [1] by John M. Lee. Many of the subsequent proofs are only slight modifications of Lee's, to account for differing (but equivalent) definitions of manifolds and their manifold-interior and manifold-boundary points.

2 Notation

- $\mathbb{R}_+$ is the set of nonnegative real numbers.
- The "upper" halfspace in $\mathbb{R}^n$ is denoted by $\mathbb{H}^n_+ := \left(\mathbb{R}^{n-1} \times \mathbb{R}_+\right)$. This is isometric to $\left\{x \in \mathbb{R}^n \mid x_n \geq 0\right\}$, and isometries are homeomorphisms, so for our purposes we won't need to distinguish the two.
- The open and closed balls of radius r around x are denoted by $B_r\left(x\right)$ and $\overline{B}_r\left(x\right)$, respectively.
- The topological interior and boundary of X are denoted by $\operatorname{int}\left(X\right)$ and $\operatorname{bdy}\left(X\right)$, respectively.

3 Manifolds

Definition of a Topological Manifold with Boundary. Let M be a topological space such that

1. the topology on M has a countable basis (M is second countable), and

2. there exist disjoint open neighborhoods of any two distinct points of M (M is Hausdorff).

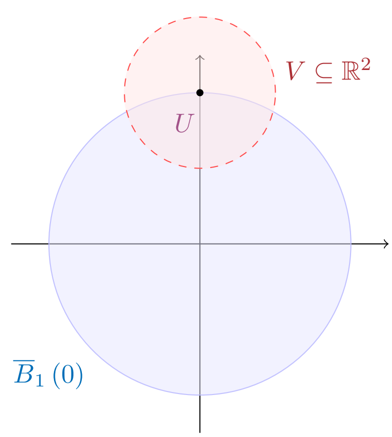
We say that M is a topological manifold of dimension n if and only if every $x \in M$ has an open neighborhood $U \ni x$ such that U is homeomorphic to an open subset of $\operatorname{int}\left(\mathbb{H}_{+}^{n}\right)$ considered as a topological subspace of $\mathbb{R}^n$. We say that M is a topological manifold of dimension n *with boundary* if and only if, in the same context, the neighborhoods U are homeomorphic to open subsets of $\mathbb{H}_{+}^{n}$.

It follows from the definition that every topological manifold of dimension n is also a topological manifold of dimension n with boundary, because open subsets of $\operatorname{int}\left(\mathbb{H}_{+}^{n}\right)$ are also open subsets of $\mathbb{H}_{+}^{n}$.

Manifolds are characterized by a *local* property. For example, the usual way to show that the unit sphere $\operatorname{bdy}\left(\overline{B}_1(0)\right)$ in $\mathbb{R}^2$ is a manifold is through the global stereographic projection that sends the sphere to $\mathbb{R}$. But that's harder than it has to be: if you zoom in close enough around any point on the unit sphere, you'll see something that looks very much like an open interval in $\mathbb{R}$. In three dimensions, this manifests as the well-known fact that the Earth is flat.

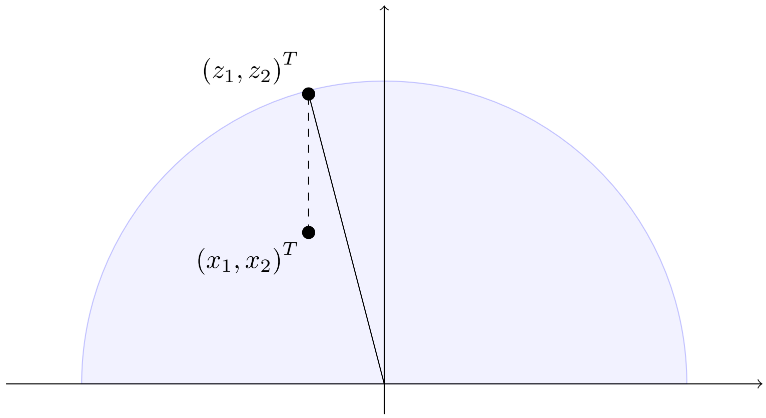
Example 1. The closed unit ball $\overline{B}_1(0)$ in $\mathbb{R}^2$ is a topological manifold of dimension two with boundary. Let x be any point in the interior of $\overline{B}_1(0)$. Then clearly, we can put an open ball U around x that is contained entirely in $\overline{B}_1(0)$. Translating U up into $\operatorname{int}\left(\mathbb{H}_{+}^{2}\right)$ is now a homeomorphism between an open neighborhood of x and an open subset of $\operatorname{int}\left(\mathbb{H}_{+}^{2}\right)$.

To deal with the boundary of $\overline{B}_1(0)$, we first make an observation: for any $x, y \in \operatorname{bdy}\left(\overline{B}_1(0)\right)$, there is an invertible linear rotation that sends x to y. Therefore by pre-composing with a rotation, it suffices to check the Definition of a Topological Manifold with Boundary at only a single point of $\operatorname{bdy}\left(\overline{B}_1(0)\right)$. For simplicity, we consider the point $(0,1)^T$. Let U be an neighborhood of $(0,1)^T$ in $\overline{B}_1(0)$, which is without loss of generality the intersection of $\overline{B}_1(0)$ and an open ball V around $(0,1)^T$.



The idea is that every point in U has some point "above" it on $\operatorname{bdy}\left(\overline{B}_1(0)\right)$, and we want to shift those points (continuously) down until they lie on $\mathbb{R}\times\{0\}$.

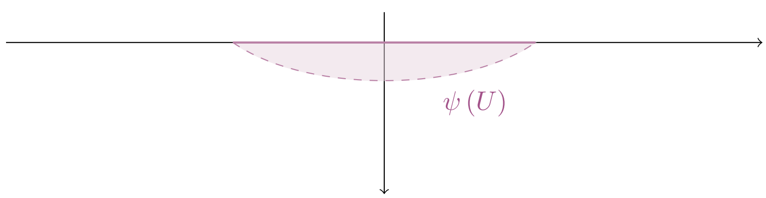
Let $(x_1,x_2)^T \in U$. The point $z=(z_1,z_2)^T$ that lies above it on $\operatorname{bdy}\left(\overline{B}_1(0)\right)$ has $z_1=x_1$, and satisfies $z_1^2+z_2^2=1$, since it's on the unit sphere. Solving for z_2 we find $z_2=\sqrt{1-x_1^2}$, which is how far down we want to shift everything in U that has $x_1=z_1$ as its first coordinate.



This leads to the map

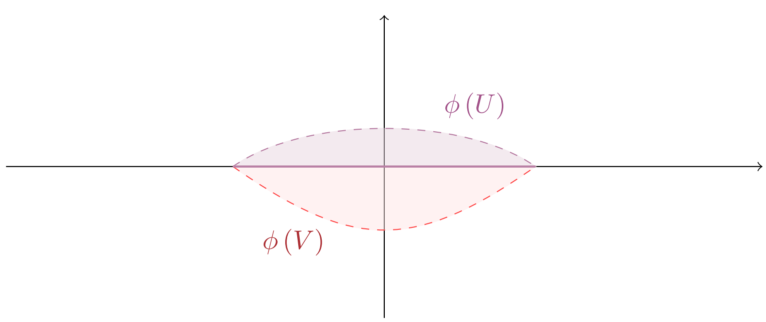
$$\psi \coloneqq (x_1,x_2)^T \mapsto \left(x_1, x_2-\sqrt{1-x_1^2}\right)^T,$$

which is a homeomorphism sending U to an open subset of $-\mathbb{H}^2_+$.



To fix the second coordinate, we just flip it, and call the resulting map ϕ:

$$\phi \coloneqq (x_1,x_2)^T \mapsto \left(x_1, \sqrt{1-x_1^2}-x_2\right)^T.$$



The map ϕ its own inverse and is clearly continuous given that $x_1 \leq 1$. If the relatively-open U was obtained as $V \cap \overline{B}_1(0)$, then ϕ maps U onto the relatively-open subset $\phi(V) \cap \mathbb{H}^2_+$.

Definition of Manifold Interior and Boundary. If M is a topological manifold (with or without boundary) of dimension n, then a point $x \in M$ is

1. a manifold-interior point of M if there exists an open $U \ni x$ and a homeomorphism ϕ such that $\phi(U)$ is open in $\operatorname{int}(\mathbb{H}^n_+)$;

2. a manifold-boundary point of M if there exists an open $U \ni x$ and a homeomorphism ϕ such that $\phi(U)$ is open in $\mathbb{H}^n_+$ and $\phi(x) \in \operatorname{bdy}(\mathbb{H}^n_+)$.

We write $\operatorname{mint}(M)$ for the set of all manifold-interior points of M, and $\operatorname{mbdy}(M)$ for its manifold-boundary points.

Beware that many authors say simply "interior" and "boundary," leaving it to the reader to infer from the context whether the topological or manifold interior and boundary are meant. Moreover, the Definition of Manifold Interior and Boundary admits the possibility of points $x \in M$ being simultaneously manifold-interior and manifold-boundary points of M. Proving that there are no such points is nontrivial, forcing us to punt.

Invariance of Boundary Theorem. *If M is a topological manifold of dimension n, then M is the disjoint union of* $\operatorname{mint}(M)$ *and* $\operatorname{mbdy}(M)$.

Lee, Theorem 2.59 and Problem 13–4

Using this result we can begin to deduce some important properties.

Proposition 1. *If M is a topological manifold of dimension n with boundary, then* $\operatorname{mint}(M)$ *is an open subset of M and is itself a topological manifold of dimension n without boundary.*

Lee, Proposition 2.58

Proof. The set $\operatorname{mint}(M)$ is open in M if and only if every $x \in \operatorname{mint}(M)$ has an open neighborhood contained in $\operatorname{mint}(M)$. Suppose that $x \in \operatorname{mint}(M)$, so that by definition x has a neighborhood U and corresponding homeomorphism ϕ with $\phi(U)$ open in $\operatorname{int}(\mathbb{H}^n_+)$. If $y \in U$ then ϕ is also a homeomorphism that takes the neighborhood U of y to an open subset of $\operatorname{int}(\mathbb{H}^n_+)$. Thus, $U \subseteq \operatorname{mint}(M)$, showing that $\operatorname{mint}(M)$ is an open set in M. And the definition of $\operatorname{mint}(M)$ more or less says that it's a topological manifold of dimension n without boundary. □

Corollary 1. *The set* $\operatorname{mbdy}(M)$*, being the complement of* $\operatorname{mint}(M)$ *by the Invariance of Boundary Theorem, is a closed subset of M.*

Lee, Corollary 2.60

This next result is sometimes given as the definition of the manifold-interior and manifold-boundary, but it's instructive to prove the strong characterization from the weak one, because we'll use the technique again.

Proposition 2. *If M is a topological manifold of dimension n with boundary, and if $U \subseteq M$ is open, then any homeomorphism $\phi : U \to \phi(U) \subseteq \mathbb{H}^n_+$ maps* $\operatorname{mint}(M)$ *to* $\operatorname{int}(\mathbb{H}^n_+)$ *and* $\operatorname{mbdy}(M)$ *to* $\operatorname{bdy}(\mathbb{H}^n_+)$.

Proof. Let $U \subseteq M$ be open, $\phi : U \to \phi(U)$ be a homeomorphism, and $x \in U$.

If $x \in \operatorname{mint}(M)$, then there exists some open neighborhood $V \ni x$ and homeomorphism $\psi : V \to \psi(V) \subseteq \operatorname{int}(\mathbb{H}^n_+)$. But then the Invariance of Boundary Theorem says that ϕ must also send U to $\operatorname{int}(\mathbb{H}^n_+)$, because the two possibilities in the Definition of Manifold Interior and Boundary are mutually exclusive.

Likewise, if $x \in \operatorname{mbdy}(M)$, then there exists some open $V \ni x$ and a homeomorphism $\psi : V \to \psi(V)$ that sends x to $\operatorname{bdy}(\mathbb{H}^n_+)$. As before, this implies that ϕ also sends one point of U to $\operatorname{bdy}(\mathbb{H}^n_+)$, because if not then it would send U to $\operatorname{int}(\mathbb{H}^n_+)$ and the Invariance of Boundary Theorem says that cannot be. We claim that $\phi(x) \in \operatorname{bdy}(\mathbb{H}^n_+)$. If not, then $\phi(x) \in \operatorname{int}(\mathbb{H}^n_+)$, and we can put an open neighborhood $W \ni \phi(x)$ around it that is disjoint from $\operatorname{bdy}(\mathbb{H}^n_+)$. And in that case, $\phi^{-1}(W)$ is an open subset of U (and thus of M); but, that would imply that the open neighborhood $\phi^{-1}(W) \ni x$ is homeomorphic to an open subset W of $\operatorname{int}(\mathbb{H}^n_+)$ under the restriction of ϕ to $\phi^{-1}(W)$. That would be a contradiction, as we would then conclude that $x \in \operatorname{mint}(M)$. As a result, we must have had $\phi(x) \in \operatorname{bdy}(\mathbb{H}^n_+)$ all along. □

Theorem 1. *If M is a topological manifold of dimension n with boundary, then $\operatorname{mbdy}(M)$ is a topological manifold of dimension $n-1$ without boundary.* *Lee, Problem 3–1*

Proof. Suppose that $x \in \operatorname{mbdy}(M)$, let $U \ni x$ be open in M, and let $\phi : U \to \phi(U) \subseteq \mathbb{H}^n_+$ be the associated homeomorphism. Proposition 2 says that ϕ maps all points of $\operatorname{mbdy}(M)$ to $\operatorname{bdy}(\mathbb{H}^n_+)$. The set $U \cap \operatorname{mbdy}(M)$ is an open neighborhood of $x \in \operatorname{mbdy}(M)$ in the subspace topology, and if we restrict ϕ to $U \cap \operatorname{mbdy}(M)$, it remains a homeomorphism whose image (necessarily open in $\mathbb{H}^n_+$) happens to be contained in $\operatorname{bdy}(\mathbb{H}^n_+)$. As a result, its image is relatively open in $\operatorname{bdy}(\mathbb{H}^n_+)$, and x has an open neighborhood in $\operatorname{mbdy}(M)$ that is homeomorphic to a relatively open subset of $\operatorname{bdy}(\mathbb{H}^n_+)$. Now we simply note that $\operatorname{bdy}(\mathbb{H}^n_+)$ is homeomorphic to $\mathbb{R}^{n-1}$ and thus to $\operatorname{int}(\mathbb{H}^{n-1}_+)$. □

4 Proper cones

Definition 1. Let V be a finite-dimensional real Hilbert space. A nonempty subset K of V is a *cone* if $\alpha K \subseteq K$ for all $\alpha \geq 0$. A cone is *proper* if it is topologically closed and convex, has nonempty interior, and does not contain any nontrivial subspaces.

Theorem 2. *If K is a proper cone in a n-dimensional real Hilbert space, then K is a topological manifold of dimension n whose topological boundary is a topological manifold (without boundary) of dimension $n-1$.*

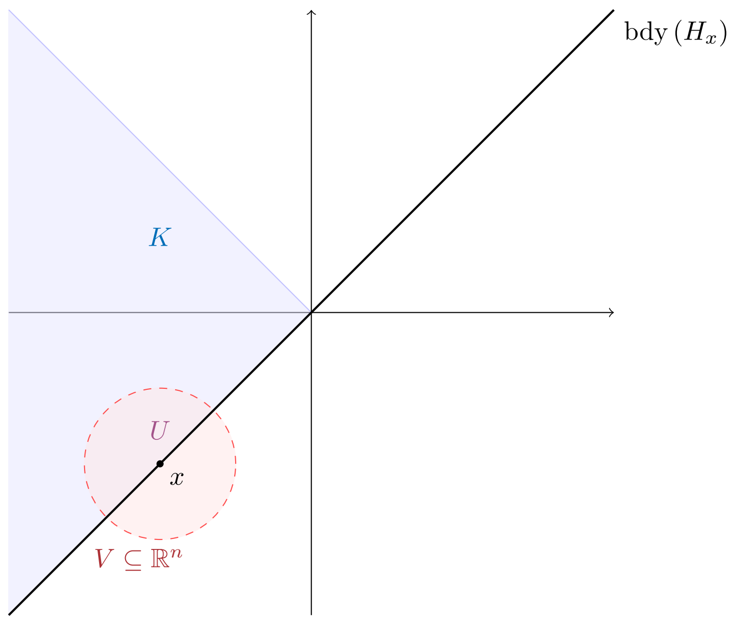
Proof. All n-dimensional real Hilbert spaces are isometric (and thus homeomorphic) to $\mathbb{R}^n$, so without loss of generality, we pretend that K lives in $\mathbb{R}^n$.

Any point $x \in \operatorname{int}(K)$ admits an open neighborhood $U \ni x$ contained entirely within K. One can take that neighborhood U and translate it until it lies in $\operatorname{int}\left(\mathbb{H}_{+}^{n}\right)$, and that's your homeomorphism. As a result, the topological interior of K is a topological manifold without boundary of dimension n.

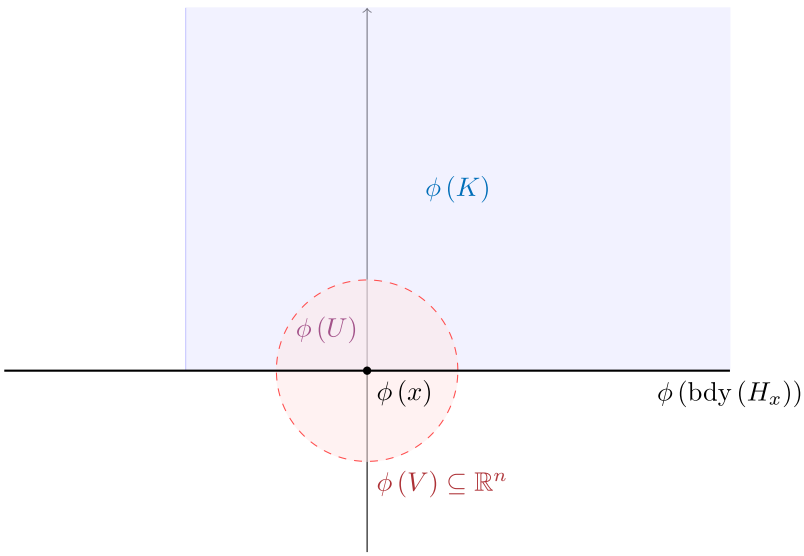
If instead we have $x \in \operatorname{bdy}(K)$, then Corollary 11.7.1 in Rockafellar [2] states that K is the intersection of the half-spaces that contain it,

$$K = \bigcap \{H \mid H \text{ is a homogeneous closed half-space in } \mathbb{R}^n \text{ and } H \supseteq K\}. \quad (1)$$

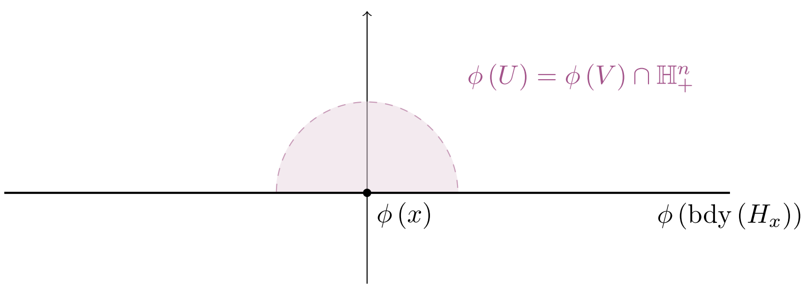
In particular our $x \in \operatorname{bdy}(K)$ must lie on the topological boundary of at least one H in Equation (1), because if not, then it lies in the topological interior of all of them and a contradiction ensues. Let H_x be any half-space in Equation (1) with x on its topological boundary, and let $U \ni x$ be a relatively-open neighborhood of x obtained by intersecting the open set $V \subseteq \mathbb{R}^n$ with K.



Now let ϕ be the map that translates x to the origin, and then reorients H_x to be the upper half-space $\mathbb{H}_{+}^{n}$.



As the composition of a translation and a change of coordinates, ϕ is a homeomorphism. Now $\phi(x) \in \text{bdy}(\mathbb{H}^n_+)$, and $\phi(U) \ni \phi(x)$ is a relatively-open neighborhood of $\phi(x)$ in $\mathbb{H}^n_+ = \phi(H_x)$.



If we restrict the domain of ϕ to U, then the resulting map remains a homeomorphism onto its image. Thus, x lies on the manifold boundary of K, and we conclude that the topological boundary of K is its manifold boundary from the Invariance of Boundary Theorem. The fact that the topological boundary of K is a manifold of dimension $n-1$ now follows from Theorem 1. □

We used the fact that K has nonempty topological interior when we intersected an open set with K and its dimension remained unchanged, but we never relied upon K containing no nontrivial subspaces. Everything still works if we omit that condition. For example if $K = \mathbb{R}^n$ is our (closed, convex, with nonempty interior) cone in $V = \mathbb{R}^n$, then $\text{bdy}(K) = \emptyset$, and the result is true because $\text{mbdy}(K) = \text{bdy}(K) = \emptyset$ and the empty set vacuously satisfies the Definition of a Topological Manifold with Boundary of any dimension.

Indeed, Rockafellar's Theorem 11.5 states that any closed convex set is the intersection of the closed half-spaces that contain it [2]. Our result thus extends to any closed convex set whose topological interior is nonempty.

Question 1. A *smooth* manifold is a topological manifold where the homeomorphisms (and their inverses) are differentiable. Are proper cones smooth manifolds with boundary as well?

Answer 1. Not in general. Any *polyhedral* proper cone will have "corners" on its boundary. A neighborhood of a corner can be continuously deformed into the flat boundary of $\mathbb{H}^n_+$, but the deformation won't be smooth.

Question 2. What additional conditions do we have to place on a proper cone to make it into a smooth manifold with boundary?

References

[1] John M. Lee. *Introduction to Topological Manifolds*, vol. 202 of *Graduate Texts in Mathematics*. Springer, New York, second ed., 2011. ISBN 9781441979401, doi:10.1007/978-1-4419-7940-7.

[2] Ralph Tyrrell Rockafellar. *Convex Analysis*. Princeton University Press, Princeton, 1970. ISBN 9780691015866.