Gaddum's test for symmetric cones

Michael Orlitzky

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Abstract

A real symmetric matrix $A \in \mathcal{S}^n$ is *copositive* if $\langle Ax, x \rangle \geq 0$ for all x in the nonnegative orthant. Copositive programming has attracted a lot of attention since Burer showed that hard nonconvex problems can be formulated as completely-positive programs. Alas, the power of copositive programming is offset by its difficulty: simple questions like "is this matrix copositive?" have complicated answers. In 1958, Jerry Gaddum proposed a recursive procedure to check if a given matrix is copositive by solving a series of matrix games. It is easy to implement and conceptually simple.

Copositivity generalizes to cones other than the nonnegative orthant. If K is a proper cone, then the linear operator L is copositive on K if $\langle L(x), x \rangle \geq 0$ for all x in K. Little is known about these operators in general. We extend Gaddum's test to self-dual and symmetric cones, thereby deducing criteria for copositivity in those settings.

1 Introduction

This is a story about *copositivity*. In the beginning, there was game theory: von Neumann and Morgenstern [38] published the "Theory of Games and Economic Behavior" in 1944, and the field of game theory was born. Shortly thereafter, Dantzig formalized the notion of a linear program and devised his simplex method to solve them. The simultaneous emergence of digital computers made it possible to solve linear programs quickly, and von Neumann [8] noticed as early as 1947 that any matrix game can be posed as a linear program.

The notion of copositivity was subsequently introduced by Motzkin [27] in 1952. A *copositive matrix* is a real, symmetric, n-by-n matrix A such that $\langle Ax, x \rangle \geq 0$ for all $x \in \mathbb{R}^n_+$. Here and hereafter, the symbol $\mathbb{R}^n_+$ denotes the componentwise-nonnegative orthant in $\mathbb{R}^n$. Originally, copositivity was studied via copositive *quadratic forms*, but every copositive quadratic form corresponds to some copositive matrix and vice-versa. The difficulty was immediately apparent; simple questions like "is this matrix copositive?" have complicated answers. A few such tests were proposed [27], but one in particular stands out. In 1958, Jerry Gaddum [18] combined game theory and linear programming to formulate the following recursive procedure.

Gaddum's test. *If $A \in \mathbb{R}^{n \times n}$ is symmetric and if each principal submatrix of A of order $(n-1)$ is copositive, then A is copositive if and only if the value of the matrix game associated with A is nonnegative.*

Here the timeline splits for a few decades. Fiedler [17] and Loewy and Schneider [31] were likely the first to extend the notion of copositivity to more difficult cones (the Lorentz "ice-cream" cone, in this case), but not until the mid-seventies. Back in the world of linear programming, progress was being made at an incredible pace, culminating in the 1984 paper of Karmarkar [30] showing that barrier-function "interior point" methods could solve linear programs in polynomial time. Karmarkar's method was extended to second-order cone programs (SOCP) and semidefinite cone programs (SDP), and another major breakthrough in conic optimization was made in 1994 when Nesterov and Nemirovskii introduced their self-concordant barrier functions [33]. The authors showed that a "universal" self-concordant barrier function exists for any open convex set, and that on the interior of a proper cone, the universal barrier function is log-homogeneous. Computing it, however, is generally problematic.

Güler used the theory of Jordan algebras to show in 1996 that this universal barrier function is essentially known for homogeneous cones [24]. He also pointed out that self-dual homogeneous cones correspond exactly to the cones of squares in Euclidean Jordan algebras. At the same time, Nesterov and Todd [34] were showing that particularly efficient methods exist for "self-scaled" conic optimization, and self-scaled cones turn out to be nothing other than self-dual homogeneous cones. Thus, the most efficient interior-point barrier methods apply to the cones that arise as cones of squares in Euclidean Jordan algebras. Over the next five years or so, Faybusovich cemented these ideas by showing that certain efficient algorithms (in particular, Nesterov-Todd) can be described directly in Jordan-algebraic terms [14, 15, 16].

The nonnegative orthant, Lorentz "ice-cream" cone, and the cone of real symmetric positive-semidefinite matrices are all examples of self-dual homogeneous cones, which we now call *symmetric cones*. Symmetric cone programming has become popular over the past two decades because many hard problems can be reformulated as symmetric cone programs. For example, the famous NP-hard traveling salesman problem has several formulations as a semidefinite program [7, 10, 9]. Since we can solve semidefinite programs with relative ease, there must be a trade-off involved: each reformulation is weaker than the true traveling salesman problem, and can produce non-optimal solutions [25]. But as these things usually go, we can get better solutions by working harder.

In the subspace $\mathcal{S}^n \subseteq \mathbb{R}^{n \times n}$ of all real symmetric n-by-n matrices, the positive-semidefinite matrices form a proper cone $\mathcal{S}^n_+$ over which semidefinite programming takes place. The set of all copositive matrices in $\mathcal{S}^n$ also forms a proper cone, but one much larger and less wieldy than $\mathcal{S}^n_+$. In 2008, Samuel Burer [4, 5] made a major breakthrough when he showed that every binary nonconvex quadratic program can be formulated as a cone program over the dual of this copositive cone. This result has rekindled the interest in copositive matrices, and brings us full circle, back to Gaddum's test.

Two threads, however, were left dangling. Game theory and linear programming were closely intertwined during their formative years. Cone progamming, on the other hand, developed directly from linear programming and convex optimization, independent of game theory. What part does game theory play in conic optimization? And what of copositivity with respect to a more general cone? We will address both of these questions.

In 2015, Gowda and Ravindran [20] introduced "linear games," which are basically two-person zero-sum games over a self-dual cone. The authors were motivated by some results in classical game theory concerning classes of matrices, and were able to extend many of those results to operators that are imporant in cone programming. Two years later, Orlitzky [35] extended those ideas to proper (but not necessarily self-dual) cones, and showed that the resulting game can be posed as a cone program. When we can solve the cone program efficiently—in particular, when the cone is symmetric—we can solve the linear game efficiently as well.

To connect linear games to copositivity, we will extend Gaddum's test to self-dual and symmetric cones, showing in the latter case that there is a recursive procedure to determine whether or not an operator is copositive on a symmetric cone. At each step, our procedure involves the solution of linear games that we now know can be solved efficiently.

2 Gaddum's test

Our game theory notation is classical and by now completely standard. The real n-fold Cartesian product space $\mathbb{R}^n$ has the usual inner-product, standard basis $\{e_1, e_2, \ldots, e_n\}$, and nonnegative orthant $\mathbb{R}^n_+ \coloneqq \{x \in \mathbb{R}^n \mid x_i \geq 0 \text{ for all } i\}$. Linear operators from $\mathbb{R}^m$ to $\mathbb{R}^n$ are represented by matrices $A \in \mathbb{R}^{n \times m}$.

Definition 1 (convex hull, conic hull)**.** In any finite-dimensional real Hilbert space V, we denote the *convex hull* of a nonempty set $X \subseteq V$ by

$$\operatorname{conv}(X) \coloneqq \left\{ \sum_{i=1}^{m} \alpha_i x_i \;\middle|\; m \in \mathbb{N}, x_i \in X, \alpha_i \geq 0, \sum_{i=1}^{m} \alpha_i = 1 \right\}.$$

The *conic hull* of X consists of all nonnegative scalar multiples of the convex hull; specifically,

$$\operatorname{cone}(X) \coloneqq \left\{ \sum_{i=1}^{m} \alpha_i x_i \;\middle|\; m \in \mathbb{N}, x_i \in X, \alpha_i \geq 0 \right\}.$$

Gaddum's test for copositivity involves solving a sequence of two-person zero-sum matrix games. These games originated with von Neumann and Morgenstern [38], but Karlin [29] provides a good introduction to the material from around the same time that Gaddum published his test. A two-person zero-sum matrix game consists of a matrix $A \in \mathbb{R}^{n \times n}$ and a compact convex set of "strategies" denoted by $\Delta \coloneqq \operatorname{conv}(\{e_1, e_2, \ldots, e_n\})$ from which the two players choose.

If the players choose x and y respectively from Δ, then the game is played by evaluating $y^T Ax$ as the "payoff" to the first player. The payoff to the second player is $-y^T Ax$, fulfilling the promise that something should sum to zero. For later reference, we remark that the strategy set Δ has an alternate description,

$$\Delta = \left\{ x \in \mathbb{R}^n_+ \;\middle|\; \langle x, e \rangle = 1 \right\}, \text{ where } e \coloneqq (1, 1, \ldots, 1)^T \in \operatorname{int}\left(\mathbb{R}^n_+\right). \tag{1}$$

An axiom of economics states that each player will try to maximize his payoff in this scenario; or, equivalently, try to minimize the payoff to his opponent. The existence of optimal strategies is guaranteed for both players, and the *value of the matrix game A* is defined to be the payoff resulting from optimal play,

$$v(A) \coloneqq \max_{x \in \Delta} \min_{y \in \Delta} \left(y^T Ax \right) = \min_{y \in \Delta} \max_{x \in \Delta} \left(y^T Ax \right).$$

The payoff to the first player in this case is $v(A)$. Necessarily corresponding to $v(A)$ is an optimal strategy pair $(\bar{x}, \bar{y}) \in \Delta \times \Delta$ such that

$$\bar{y}^T Ax \leq v(A) = \bar{y}^T A\bar{x} \leq y^T A\bar{x} \text{ for all } (x, y) \in \Delta \times \Delta. \tag{2}$$

The existence of optimal strategies and this "saddle-point" inequality follow from Karlin's Theorems 1.3.1 and 1.5.1 which are much more general statements about continuous functions on compact convex sets [29]. Copositivity becomes important here for a simple reason that constitutes the "easy direction" in Gaddum's theorem.

Proposition 1. *If $A \in \mathcal{S}^n$ is copositive, then $v(A) \geq 0$.*

Proof. Let $(\bar{x}, \bar{y}) \in \Delta \times \Delta$ be an optimal strategy pair. Then $x \coloneqq \bar{y}$ would be a valid strategy for the first player as well. Moreover, the entire strategy set Δ is contained in the nonnegative orthant. Thus if the game matrix A is copositive, the Inequality (2) gives $v(A) \geq \bar{y}^T Ax = \bar{y}^T A\bar{y} \geq 0$. ☐

Naturally the converse does not hold: there are plenty of non-copositive matrices whose games have a nonnegative value. We recall Gaddum's Theorem 3.2 [18] to complete the characterization, but we borrow our presentation of the theorem from Hiriart-Urruty and Seeger's survey article [27] where it is stated more conveniently.

Theorem 1 (Gaddum's test)**.** *If $A \in \mathcal{S}^n$ and if each principal submatrix of A of order $(n-1)$ is copositive, then A is copositive if and only if $v(A) \geq 0$.*

Note that "principal submatrix" here refers to one where the *same* column and row indices have been deleted; in Gaddum's original paper, the corresponding operation is to set certain components of x equal to zero in the quadratic form $x \mapsto \langle Ax, x \rangle$. The fact that we need only consider submatrices of order $(n-1)$ can be inferred from Gaddum's proof, since every boundary point of $\mathbb{R}^n_+$ lies in some $(n-1)$-dimensional face of $\mathbb{R}^n_+$.

The practical appeal of this result is that the value of a matrix game can be found by solving a linear program [8], and linear programs of enormous size are solved instantaneously using modern algorithms and machines. If you look closely, Theorem 1 describes a recursive procedure. If $A \in \mathcal{S}^n$, then verifying the conditions of Theorem 1 involves checking the copositivity of matrices in $\mathcal{S}^{(n-1)}$. This will involve solving a linear program whose size is on the order of $(n-1)$, and then checking the copositivity of matrices in $\mathcal{S}^{(n-2)}$, and so on. There are many such problems to solve, but each one is tractable.

3 For self-dual cones

We now need to introduce a few concepts that can be defined in quite some generality. However, since we will ultimately wind up in a finite-dimensional Euclidean Jordan algebra, and since those algebras are over the real field and are equipped with an inner-product, we won't venture outside of that setting. So from now on, even if a definition could be stated more generally, we do so in a finite-dimensional real Hilbert space.

Fortunately, much of the terminology in these spaces is standard. If V is a finite-dimensional real Hilbert space, then $\mathcal{B}(V)$ is the space of all linear operators on V. The inner-product on V induces a norm, which induces a metric, which induces a topology, with respect to which we can talk about the interior $\operatorname{int}(X)$ and boundary $\operatorname{bdy}(X)$ of a set $X \subseteq V$.

Definition 2 (cones, faces)**.** In a finite-dimensional real Hilbert space V, a nonempty subset K of V is a *cone in* V if $\alpha K \subseteq K$ for all $\alpha \geq 0$. A *closed convex cone in* V is a cone that is closed and convex as a subset of V. A convex cone K in a finite-dimensional real Hilbert space V is *solid* if $\dim(\operatorname{span}(K)) = \dim(V)$, and *pointed* if $\dim(-K \cap K) = 0$. A pointed, solid, closed convex cone is *proper*. A subcone F of K is called a *face* of K if $x, y \in K$ and $x + y \in F$ together imply that $x, y \in F$. We indicate that F is a face of K by writing $F \trianglelefteq K$. A *proper face* of K is a face F of K that is not equal to K, and is denoted by $F \triangleleft K$.

Since each face of a cone is itself a cone, the phrase "proper face" must be wielded with caution. When we say "proper face" of a "proper cone," we mean that the big cone is solid, pointed, et cetera—and that the face is not equal to it. Whether or not the proper face is a proper cone in its own right is a separate question. Perhaps the best way to think about this is to not think about this.

The faces of a proper cone form a complete lattice of finite length with respect to the "is a face of" ordering [1]. Each point in the cone belongs to the face generated by that point, and therefore to some maximal face with respect to the lattice ordering. As a special case of Dickinson's Theorem 2.8, we see that the boundary of a proper cone is the union of its maximal proper faces [11]. We record this fact and an immediate corollary for later reference.

Proposition 2. *If K is a proper cone in a finite-dimensional real Hilbert space, then the boundary of K is the union of its maximal proper faces.*

Corollary 1. *If K is a proper cone in a finite-dimensional real Hilbert space and if $x \in \operatorname{bdy}(K)$, then x belongs to some maximal proper face of K.*

Definition 3 (dual cone)**.** If K is a cone in a finite-dimensional real Hilbert space V, then the *dual cone of* K *in* V is

$$K^{*} \coloneqq \{y \in V \mid \langle x, y \rangle \geq 0 \text{ for all } x \in K\}.$$

If $K = K^{*}$ then K is a *self-dual cone in* V.

Ben-Israel's Theorem 1.3 shows that K^{*} is in fact a closed convex cone [3], which justifies calling it the "dual cone of K." Rockafellar's Theorem 14.6, reproduced in the following proposition, evinces the duality between pointed and solid cones [37]. From this duality and the fact that K^{*} is a closed convex cone, we infer that every self-dual cone is proper.

Proposition 3. *If K is a closed convex cone, then* $-K \cap K = \operatorname{span}(K^{*})^{\perp}$.

Definition 4 (copositive operator)**.** If K is a cone in a finite-dimensional real Hilbert space V and if $L \in \mathcal{B}(V)$ satisfies

$$\forall x \in K : \langle L(x), x \rangle \geq 0,$$

then L is *copositive* on K.

Now that we know a bit about cones, we can introduce the linear games of Gowda and Ravindran [20]. These games will be unfamiliar to most readers, but the only parts of the theory that we're going to use are those that arise directly by analogy with the classical case. For the etymologically curious, the name "linear game" appears only in a later work by Gowda [19].

Definition 5 (linear game)**.** A *linear game* $\mathcal{G} \coloneqq (L, K, e)$ consists of a linear operator L on a finite-dimensional real Hilbert space V, a self-dual cone K in V, and a point e in the interior of K. Associated to every game is a *strategy set*

$$\Delta \coloneqq \{x \in K \mid \langle x, e \rangle = 1\}$$

and a *payoff function* $(x, y) \mapsto \langle L(x), y \rangle$ defined on $\Delta \times \Delta$.

Since we will never have more than one game in scope at the same time, we suppress the dependence of Δ on both K and e for notational convenience. As in the classical case, we imagine two players choosing strategies x and y respectively from Δ. The game is then played by evaluating the payoff function at (x, y), and by paying the amount $\langle L(x), y \rangle$ to the first player out of the second player's pocket.

The construction of the strategy set Δ in Definition 5 is by analogy with Equation (1) in the classical case. Since the point e lies in the interior of K, and since K is K^{*}, the strategy set Δ is guaranteed to be compact and convex [20]. And of course, the payoff function in a linear game is entirely equivalent to the classical one. These choices were made to ensure that linear games do not escape the purview of Karlin's Theorems 1.3.1 and 1.5.1 that guaranteed the existence of optimal strategies and a saddle-point inequality in the classical case [29].

Definition 6 (value of a linear game, optimal strategies)**.** If $\mathcal{G} \coloneqq (L, K, e)$ is a linear game, then the *value of* $\mathcal{G}$ is

$$v(\mathcal{G}) \coloneqq \max_{x \in \Delta} \min_{y \in \Delta} \langle L(x), y \rangle = \min_{y \in \Delta} \max_{x \in \Delta} \langle L(x), y \rangle,$$

and any $(\bar{x}, \bar{y}) \in \Delta \times \Delta$ satisfying

$$\langle L(x), \bar{y} \rangle \leq v(\mathcal{G}) = \langle L(\bar{x}), \bar{y} \rangle \leq \langle L(\bar{x}), y \rangle \text{ for all } (x, y) \in \Delta \times \Delta \tag{3}$$

is called an *optimal strategy pair* for $\mathcal{G}$.

As you would expect, when $K = \mathbb{R}^n_+$ and $e = (1, 1, \ldots, 1)^T$, a linear game reduces to a two-person zero-sum matrix game. Orlitzky showed that the players in a linear game are trying to solve dual cone programs, and that therefore every linear game can be posed as a cone program [35]. That correspondence, like this next result that appears as Proposition 1 in Gowda and Ravindran [20], is once more completely analogous to the classical case.

Proposition 4. *If $\mathcal{G} \coloneqq (L, K, e)$ is a linear game and if L is copositive on K, then $v(\mathcal{G}) \geq 0$.*

Proof. As in Proposition 1, the definition of copositivity and $\Delta \subseteq K$ together imply that $v(\mathcal{G}) \geq \langle L(\bar{y}), \bar{y} \rangle \geq 0$ in Inequality (3). $\square$

The last thing that we need to state our main results is the notion of a principal subtransformation. This we borrow from Gowda and Tao [23], who studied them in relation to the cone complementarity problem.

Definition 7 (orthogonal projections, principal subtransformations)**.** If V is a finite-dimensional real Hilbert space and if $X \subseteq V$, then π_X denotes the self-adjoint linear orthogonal projection onto the subspace $\operatorname{span}(X)$ of V. If K is a closed convex cone in V, if $L \in \mathcal{B}(V)$, and if F is a face of K, then *the principal subtransformation of L corresponding to F* is

$$\begin{aligned} L_F &: \operatorname{span}(F) \to \operatorname{span}(F) \\ L_F &\coloneqq \pi_F \circ L. \end{aligned}$$

Principal subtransformations are intended to generalize principal submatrices, and this definition reduces to what you'd expect for faces of $\mathbb{R}^n_+$.

Example 1. Let $V = \mathbb{R}^n$, $K = \mathbb{R}^n_+$, and $L \in \mathbb{R}^{n \times n}$. It's easy to see that every face F of K is isomorphic to $\mathbb{R}^m_+$ in $\mathbb{R}^m$, where $m \coloneqq \dim(F) \leq n$. As a result, every principal subtransformation L_F of L is a principal submatrix of L.

Lemma 1. *If K is a self-dual cone in a finite-dimensional real Hilbert space, if $e \in \operatorname{int}(K)$, and if $\Delta = \{x \in K \mid \langle x, e \rangle = 1\}$, then for all $\bar{x} \in \Delta$ and for all $x \in K$ there exist*

- *a $\lambda \in [0, 1]$,*

- *an* $\alpha \geq 0$*,*
- *a maximal proper face* F *of* K*, and*
- *an element* x' *in* F

such that $x = \lambda\alpha\bar{x} + (1-\lambda)\,x'$*.*

Proof. If $x \in \operatorname{bdy}(K)$, then we can take $\alpha = \lambda = 0$, $x' = x$, and F to be the maximal proper face of K that contains x by Corollary 1—and we are done.

If $x \in \operatorname{int}(K)$, then $\langle x, e\rangle > 0$ and we can scale to obtain $x / \langle x, e\rangle \in \Delta$. Since $\bar{x} \in \Delta$ as well, the entire segment $\operatorname{conv}(\{\bar{x}, x/\langle x, e\rangle\})$ is contained in Δ. By extending that segment past $x/\langle x, e\rangle$, we eventually leave Δ which is compact and therefore bounded. Along the way, we encounter some boundary point $b \in \operatorname{bdy}(\Delta)$ such that $x/\langle x, e\rangle$ lies in the segment $\operatorname{conv}(\{\bar{x}, b\})$. Thus,

$$\exists \lambda \in [0,1] : x/\langle x, e\rangle = \lambda\bar{x} + (1-\lambda)\,b.$$

Let $\alpha = \langle x, e\rangle$, $x' = \langle x, e\rangle\, b \in \operatorname{bdy}(K)$, and F be the maximal proper face containing x' by Corollary 1. □

Theorem 2 (Gaddum's test for self-dual cones)**.** *If* K *is a self-dual cone in a finite-dimensional real Hilbert space* V *and if* $L \in \mathcal{B}(V)$*, then the following are equivalent:*

- L *is copositive on* K*.*
- *The value of the linear game* $\mathcal{G} \coloneqq (L, K, e)$ *is nonnegative, and the principal subtransformation* L_F *is copositive on* F *for all maximal* $F \triangleleft K$*.*

Proof. If L is copositive on K, then Proposition 4 shows that $v(\mathcal{G}) \geq 0$. Moreover, if L is copositive on K, then every L_F must be copositive on F; otherwise, there would exist an $x \in F$ such that $\langle L_F(x), x\rangle < 0$, and since $x \in F \subseteq K$ we could simply substitute:

$$\langle \pi_F(L(x)), x\rangle = \langle L(x), \pi_F(x)\rangle = \langle L(x), x\rangle < 0.$$

Thus, when L is copositive, it follows that every L_F is copositive and that the value of the game $\mathcal{G}$ is nonnegative.

On the other hand, if $v(\mathcal{G}) \geq 0$ with an optimal pair $(\bar{x}, \bar{y})$, then we can use Lemma 1 to write any $x \in K$ as

$$x = \lambda\alpha\bar{x} + (1-\lambda)\,x'$$

for $\lambda \in [0,1]$, $\alpha \geq 0$, $F \triangleleft K$ maximal, and $x' \in F$. Now expand using bilinearity,

$$\langle L(x), x\rangle = \lambda^2\alpha^2 \langle L(\bar{x}), \bar{x}\rangle + 2\lambda(1-\lambda)\,\alpha \langle L(\bar{x}), x'\rangle + (1-\lambda)^2 \langle L(x'), x'\rangle\,.$$

Since $(\bar{x}, \bar{y})$ is optimal for the game $\mathcal{G}$ with $v(\mathcal{G}) \geq 0$, it follows that

$$\lambda^2\alpha^2 \langle L(\bar{x}), \bar{x}\rangle \geq \lambda^2\alpha^2 \langle L(\bar{x}), \bar{y}\rangle = \lambda^2\alpha^2 v(\mathcal{G}) \geq 0.$$

For the same reason, we can conclude that $2\lambda(1-\lambda)\,\alpha\,\langle L(\bar{x}), x'\rangle \geq 0$. Thus the first two terms in the expansion of $\langle L(x), x\rangle$ are nonnegative. It remains only to show that $\langle L(x'), x'\rangle \geq 0$. However, since $\pi_F(x') = x' \in F$,

$$\langle L(x'), x'\rangle = \langle L(x'), \pi_F(x')\rangle = \langle \pi_F(L(x')), x'\rangle = \langle L_F(x'), x'\rangle .$$

And, since $x' \in F$ and L_F is copositive on F, we have $\langle L_F(x'), x'\rangle \geq 0$. It follows that $\langle L(x), x\rangle \geq 0$ as desired. □

One might wonder why we have opted for "maximal proper face" in the statement of Theorem 2, when a face of codimension one (a *facet*) would be more in line with Theorem 1. The reason is simple: facets may not exist.

Example 2. The Lorentz "ice-cream" cone in $\mathbb{R}^n$ is

$$\mathcal{L}^n_+ \coloneqq \left\{ (t, x)^T \in \mathbb{R} \times \mathbb{R}^{n-1} \;\middle|\; \|x\| \leq t \right\}.$$

It is well-known that the nonzero proper faces of $\mathcal{L}^n_+$ are its one-dimensional boundary rays. When $n > 2$, it therefore has no facets.

Since there are no intermediate proper faces between $\mathcal{L}^n_+$ and its boundary rays, a single application of Theorem 2 reduces the problem of copositivity on $\mathcal{L}^n_+$ to that of copositivity on $\operatorname{bdy}\left(\mathcal{L}^n_+\right)$. There are many important properties of an operator with respect to a proper cone that need only be checked on the boundary of that cone. For example, the operator $L \in \mathcal{B}(V)$ is a Lyapunov-like, **Z**, or positive operator on a proper cone K in V if and only if L satisfies the respective property on the boundary of K [36]. However, copositivity is not one of those properties. Suppose that a point x in some proper cone K is a conic combination of two boundary points of K; that is, $x = x_1 + x_2$ for $x_1, x_2 \in \operatorname{bdy}(K)$. Then

$$\langle L(x), x\rangle = \langle L(x_1), x_1\rangle + \langle L(x_1), x_2\rangle + \langle L(x_2), x_1\rangle + \langle L(x_2), x_2\rangle .$$

Even if we know that the two terms $\langle L(x_1), x_1\rangle$ and $\langle L(x_2), x_2\rangle$ are nonnegative here, that doesn't tell us anything about the other two terms. Thus, knowing that L is copositive on the boundary of K does not generally imply that L is copositive on all of K.

But as luck would have it, the situation with $\mathcal{L}^n_+$ turns out to be typical. And while Theorem 2 was stated in terms of maximal proper faces for superficial parity with Gaddum's original test, for visceral impact we include the additional equivalent condition in terms of copositivity on the boundary.

Lemma 2. *If K is a proper cone in a finite-dimensional real Hilbert space V and if $L \in \mathcal{B}(V)$, then the following are equivalent:*

- *L_F is copositive on F for all maximal $F \trianglelefteq K$.*
- *L is copositive on the boundary of K.*

Proof. Both implications follow easily from Proposition 2 and the identity

$$\langle L_F(x), x\rangle = \langle \pi_F(L(x)), x\rangle = \langle L(x), \pi_F(x)\rangle = \langle L(x), x\rangle. \qquad \square$$

Corollary 2. *If K is a self-dual cone in a finite-dimensional real Hilbert space V and if $L \in \mathcal{B}(V)$, then the following are equivalent:*

- *L is copositive on K.*
- *The value of the linear game $\mathcal{G} := (L, K, e)$ is nonnegative, and L is copositive on the boundary of K.*

Proof. Combine equivalent conditions in Theorem 2 and Lemma 2. $\square$

4 For symmetric cones

Unfortunately, Theorem 2 is not the end of the story. The practical appeal of Theorem 1 is as a recursive procedure: checking the copositivity of each principal submatrix is conceptually equivalent to checking the copositivity of the original matrix, sans a particular row and column. If K is a self-dual cone and if F is a maximal proper face of K, then F may look nothing like K. How, then, should we test the copositivity of L_F? Is F even self-dual? In general, no.

Example 3. In the inner-product space $\mathbb{R}^3$, let

$$\begin{gathered} v_0 := (1,1,1)^T,\ v_1 := (0,1,1)^T,\ v_2 := (-1,0,1)^T, \\ v_3 := (0,-1,1)^T,\ v_4 := (1,-1,1)^T, \\ K := \operatorname{cone}(\{v_0, v_1, v_2, v_3, v_4\}). \end{gathered}$$

Barker and Foran [2] show that this polyhedral cone is self-dual; yet if we consider the face generated by, say, v_3 and v_4, then $\operatorname{cone}(\{v_3, v_4\})$ is not self-dual in the space $\operatorname{span}(\{v_3, v_4\})$.

This example shows that we cannot necessarily apply Theorem 2 to the subtransformations L_F that we obtain from a general self-dual cone. Does this preclude the use of Theorem 2 in a recursive procedure? There are some well-known self-dual cones whose faces are isomorphic to lower-dimensional versions of themselves; we have already encountered this phenomenon with the Lorentz "ice-cream" cone in Example 2. The faces of any ice-cream cone are its one-dimensional boundary rays, and a ray is somewhat obviously a one-dimensional ice-cream cone. The same sort of thing happens with the cone of real symmetric positive-semidefinite matrices $\mathcal{S}^n_+$, which is self-dual in $\mathcal{S}^n$.

Example 4 (Hill and Waters [26], Theorems 3.4 and 3.6)**.** The set $W \subseteq \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if and only if

$$F_W := \left\{ X \in \mathcal{S}^n_+ \;\middle|\; X(\mathbb{R}^n) \subseteq W \right\}$$

is a face of $\mathcal{S}^n_+$ in $\mathcal{S}^n$. Moreover, every face $F_W \trianglelefteq \mathcal{S}^n_+$ is isomorphic to $\mathcal{S}^r_+$ for some $r \in \{0, 1, 2, \ldots, n\}$.

So we have three examples of cones whose faces are self-dual in an appropriate subspace. There's the real symmetric PSD cone of Example 4 and the Lorentz cone of Example 2, of course. But we also have our primeval Example 1. The fact that the faces of $\mathbb{R}^n_+$ look like $\mathbb{R}^m_+$ for $m \le n$ is precisely why Gaddum's original test is so elegant. These three cones are well-known examples of *symmetric* cones [13], which Güler recognized in 1996 as the cones of squares in Euclidean Jordan algebras [24]. A symmetric cone is a self-dual cone that has one additional property called homogeneity. We include the definition of homogeneity only as a prerequisite for an honest definition of a symmetric cone—it will not be needed.

Definition 8 (symmetric cone)**.** A cone K in a finite-dimensional real Hilbert space V is *homogeneous* if for all x and y in the interior of K, there exists an invertible $L \in \mathcal{B}(V)$ such that $L(K) = K$ and $L(x) = y$. If K is both self-dual and homogeneous, then it is *symmetric*.

One consequence of self-duality is that every symmetric cone is proper. The reader should be aware that the definitions of "homogeneous" and "symmetric" differ slightly between sources. For example, Faraut and Korányi's symmetric cones [13] are the interiors of what we call a symmetric cone. Our terminology follows Güler [24] and Faybusovich [14] instead.

Perhaps all symmetric cones have faces that are self-dual in their spans? This is our motivation for introducing Euclidean Jordan algebras, and the answer will be affirmative: every face of a symmetric cone in a Euclidean Jordan algebra is itself a symmetric cone in an appropriate subalgebra. For the sake of consistency, we define a Euclidean Jordan algebra to be both finite-dimensional and unital, in agreement with our main references.

Definition 9. A *Euclidean Jordan algebra* $(V, \circ, \langle\cdot,\cdot\rangle)$ consists of a finite-dimensional real Hilbert space $(V, \langle\cdot,\cdot\rangle)$ and a commutative bilinear algebra "multiplication" operation $\circ$ such that

$$\forall x, y \in V : x \circ ((x \circ x) \circ y) = (x \circ x) \circ (x \circ y),$$

$$\forall x, y, z \in V : \langle x \circ y, z\rangle = \langle y, x \circ z\rangle,$$

and having a multiplicative identity element $1_V \in V$ such that

$$\forall x \in V : 1_V \circ x = x.$$

One can think of Euclidean Jordan algebras as generalizing the real symmetric matrices. This is a crude interpretation, but with $\mathcal{S}^n$ in mind, the next definition based on Chapter III of Faraut and Korányi [13] should be reminiscent of the orthogonal projections that arise in the matrix spectral decomposition.

Definition 10 (Jordan frame)**.** If $(V, \circ, \langle\cdot,\cdot\rangle)$ is a Euclidean Jordan algebra, then $c \in V$ is *idempotent* if $c \circ c = c$. Two idempotents $c_1, c_2 \in V$ are said to be *orthogonal* if $c_1 \circ c_2 = 0$, since this implies orthogonality with respect to the inner-product. A nonzero idempotent c is *primitive* if there do not exist two nonzero idempotents c_1 and c_2 in V such that $c = c_1+c_2$. The set $\{c_1, c_2, \ldots, c_r\}$ is a *Jordan frame* if its elements are pairwise-orthogonal primitive idempotents that sum to the identity element of V.

Definition 11 (Peirce unit subalgebra)**.** If $(V, \circ, \langle\cdot,\cdot\rangle)$ is a Euclidean Jordan algebra and if J is a subset of a Jordan frame $\{c_1, c_2, \ldots, c_r\}$, then we define

$$c_J \coloneqq \sum_{c_j \in J} c_j$$

so that the linear operator $x \mapsto c_J \circ x$ has an eigenspace

$$V_J \coloneqq \{x \in V \mid c_J \circ x = x\}$$

corresponding to the eigenvalue one. We also define the set of squares on V_J,

$$K_J \coloneqq \{x \circ x \mid x \in V_J\}.$$

As $V_J \subseteq V$, we denote the associated inclusion embeddings by $\iota_J : V_J \hookrightarrow V$.

Since the set J in Definition 11 consists of orthogonal idempotents, the element c_J is itself idempotent. After restricting the algebra and inner-products, the subspace V_J therefore forms a subalgebra $(V_J, \circ_J, \langle\cdot,\cdot\rangle_J)$ by Proposition IV.1.1 of Faraut and Korányi [13]. The set K_J must then be the symmetric cone of squares in the Euclidean Jordan algebra $(V_J, \circ_J, \langle\cdot,\cdot\rangle_J)$.

The following theorem is half of Gowda and Sznajder's Theorem 3.1 [21]. The omission of the face $F = \{0\}$ is of no consequence here, since $\{0\}$ is trivially a symmetric cone in the ambient space $\operatorname{span}(F) = \{0\}$.

Theorem 3. *If $(V, \circ, \langle\cdot,\cdot\rangle)$ is a Euclidean Jordan algebra with cone of squares K and if $F \trianglelefteq K$ is a nonzero face of K, then there exists a Jordan frame $\{c_1, c_2, \ldots, c_r\}$ for V and a subset $J \subseteq \{c_1, c_2, \ldots, c_r\}$ such that $F = \iota_J(K_J)$. In other words, F is a symmetric cone in* $\operatorname{span}(F)$*.*

Before we conclude, we should probably say something about the point e that appears in both Lemma 1 and Theorem 2. There is a good reason why we have not mentioned it up until this point. Gowda later investigated the effect that this interior point has on a linear game [19], but in the original work [20] on which Theorem 2 is based, the interior point is largely irrelevant. For Theorem 2, any interior point will do.

Nevertheless, there are some low-hanging fruit nearby. Suppose we are given a symmetric cone K in some real Hilbert space and a linear operator L on that space. Before we can apply Theorem 2, we first need to find an $e \in \operatorname{int}(K)$ that we can use to construct the game $\mathcal{G} \coloneqq (L, K, e)$. Then to apply the theorem recursively would involve a series of subcones $F \triangleleft K$ with associated principal subtransformations L_F and points $e_F \in \operatorname{int}(F)$. At least in principle, we know how to obtain the subtransformations L_F and subcones $F \triangleleft K$. Is there a simple procedure for finding an $e_F \in \operatorname{int}(F)$? Fortunately, in a Euclidean Jordan algebra V, it is known that the multiplicative identity element 1_V belongs to the interior of K (Theorem III.2.I in Faraut and Korányi mentions this [13]). And it turns out that the same projection operator π_F used to construct L_F can be used to find an interior point $e_F \coloneqq \pi_F(1_V)$. This next result is well-known, and is a straightforward consequence of our definitions in any case. Gowda and Sznajder use the first part of it in the proof of their Theorem 6.2 [21].

Proposition 5. *Let $(V, \circ, \langle\cdot,\cdot\rangle)$ be a Euclidean Jordan algebra and let J be a subset of some Jordan frame $\{c_1, c_2, \ldots, c_r\}$ in V. If c_J and V_J are as in Definition 11, then $c_J = 1_{V_J} = \pi_{V_J}(1_V)$.*

The point obtained in Proposition 5 is an attractive choice. When $\langle\cdot,\cdot\rangle$ is the canonical trace inner-product $(x, y) \mapsto \operatorname{trace}(x \circ y)$, Proposition IV.3.2 of Faraut and Korányi [13] tells us that the strategy set Δ will be the convex hull of the set of primitive idempotents in the algebra. Moreover, this choice upholds the motivating analogy with the classical case. Recall from Equation (1) that the interior point in every two-person zero-sum matrix game was a vector of ones. When we move to a subspace in Theorem 1, the new interior point is simply a shorter vector of ones. It is well-known that $\mathbb{R}^n$ with the usual inner-product and componentwise multiplication forms a Euclidean Jordan algebra [13] whose identity element is $1_{\mathbb{R}^n} = (1, 1, \ldots, 1)^T$ and whose symmetric cone of squares is $\mathbb{R}^n_+$. Taking $e_F = \pi_F(1_{\mathbb{R}^n})$ for each maximal proper face F in that scenario results in the same procedure, giving us a vector of $(n-1)$ ones and ensuring that the classical algorithm is a special case of the forthcoming result.

Theorem 4 (Gaddum's test for symmetric cones)**.** *If $(V, \circ, \langle\cdot,\cdot\rangle)$ is a Euclidean Jordan algebra with symmetric cone K and if $L \in \mathcal{B}(V)$, then the following are equivalent:*

- *L is copositive on K.*
- *The value of the linear game $\mathcal{G} \coloneqq (L, K, 1_V)$ is nonnegative, and the principal subtransformation L_F is copositive on F for all maximal $F \trianglelefteq K$.*

In this equivalence, each $F \trianglelefteq K$ arises as the symmetric cone of squares K_F in a Euclidean Jordan subalgebra V_F of V such that $L_F \in \mathcal{B}(V_F)$ and $\pi_F(1_V) = 1_{V_F} \in \operatorname{int}(K_F)$. In addition, the value of the game $\mathcal{G}$ can be found by solving a symmetric cone program.

Proof. Corollary 17 in Orlitzky [35] shows that we can solve the game $\mathcal{G}$ by solving a cone program involving the symmetric cone K. Everything else follows directly from Theorem 2, Theorem 3, and Proposition 5. ☐

And of course, there is a related special case of Corollary 2.

Corollary 3. *If $(V, \circ, \langle\cdot,\cdot\rangle)$ is a Euclidean Jordan algebra with symmetric cone K and if $L \in \mathcal{B}(V)$, then the following are equivalent:*

- *L is copositive on K.*

- *The value of the linear game $\mathcal{G} := (L, K, 1_V)$—which can be found by solving a symmetric cone program—is nonnegative, and L is copositive on the boundary of K.*

We caution that the word "recursive" in our prose must be interpreted carefully. At each step, the dimension of the faces under consideration decreases by at least one, and the zero-dimensional base case is indeed trivial. However, other aspects of the problem can change significantly as the dimension decreases monotonically. Our three Examples 1, 2 and 4 where the faces all look like smaller versions of the original cone were misleading in this regard.

Every symmetric cone is, up to order, a unique orthogonal direct sum of the cones of squares in simple subalgebras. And up to isomorphism, there are only five types of them. Theorem 5 in Gowda, Sznajder, and Tao [22] is likely the most digestable reference for this fact. But if

$$K = \mathbb{R}^n_+ \oplus \mathcal{L}^n_+ \oplus \mathcal{S}^n_+,$$

then there's no reason to think that a proper face $F \trianglelefteq K$ must have the form

$$F \cong \mathbb{R}^m_+ \oplus \mathcal{L}^m_+ \oplus \mathcal{S}^m_+$$

for some $m < n$. In other words, some maximal proper faces of K may not "look like" K itself. There is also the problem of cardinality. In Gaddum's original test, the number of faces is finite and decreases at each step. Unless by chance you encounter a face that is isomorphic to $\mathbb{R}^m$ in Theorem 4, that generally won't be the case for a symmetric cone. So, you should probably not plan to enumerate the maximal proper faces of your cone one-at-a-time.

5 Conclusions

Up until now, the fact that we could solve a linear game via cone programming was a solution in search of a problem. But we see now that it has a role to play in detecting copositivity. This is attractive to optimizers because copositivity is important to the solution analysis of many convex optimization problems. Section 2.5 in Facchinei and Pang [12], for example, discusses how copositivity applies to the variational inequality and complementarity problems.

More recently, Németh and Gowda [32] described the cone of **Z**-operators on the Lorentz cone $\mathcal{L}^n_+$ in terms of the cone of operators copositive on $\mathcal{L}^n_+$. Iusem and Lara [28] found sufficient conditions for the existence of solutions to a *mixed* variational inequality problem that involve copositivity. The concept of a **Z**-operator can be traced back to the linear complementarity problem [6], so these ideas are related, and the ongoing research shows that copositivity continues to be important to complementarity problems.

Despite our cautionary remarks, we are optimistic that Theorem 4 can be of practical use. The connection to copositivity on the boundary is theoretically interesting. And ultimately, the state of the art is such that numerical evidence of copositivity is valuable. One has always been free to pick some points x from a cone and to look for counterexamples by checking if $\langle L(x), x \rangle \geq 0$. And while Theorem 4 doesn't allow us to obtain a "yes" answer by brute-force computation, Corollary 3 does show that we can perform a single computation at each step to rule out the interior points as counterexamples. Since most of a proper cone is made up of interior points, doing so provides stronger evidence for copositivity than a completely random search would.

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