Lyapunov Rank and Perfect Cones

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Let K be a proper cone in $\mathbb{R}^n$ and denote its dual by K^*.

Definition (dual cone).

The dual K^* of K is defined to be,

$$K^* := \{y \in \mathbb{R}^n : \forall x \in K, \langle x, y \rangle \geq 0\}$$

Cones and their duals are generalizations of vector (sub)spaces and their orthogonal complements.

Definitions

Definition (complementarity set).

The complementarity set of a cone K is,

$$C(K) := \{(x, s) : x \in K, s \in K^*, \langle x, s \rangle = 0\}$$

The complementarity set can be used to generalize certain optimization problems posed over "easy" cones like $K = K^* = \mathbb{R}^n_+$.

Definitions

Definition (Lyapunov-like transformation).

We say that a transformation $L \in \mathbb{R}^{n \times n}$ is *Lyapunov-like* on a proper cone K if,

$$(x, s) \in C(K) \implies \langle Lx, s \rangle = 0$$

In other words, Lx and s are perpendicular for all pairs (x, s) in the complementarity set of K.

Definition (Lyapunov rank).

For a given proper cone K, it is easy to see that the set of all Lyapunov-like transformations form a vector space, denoted $LL(K)$.

The dimension of this vector space is called the *Lyapunov rank* of K, and is denoted $\beta(K)$. (The beta refers to “bilinearity rank,” which is a synonym for the Lyapunov rank.)

The LCP

For an example, we turn to the Linear Complementarity Problem, or LCP. The LCP asks us to find a vector x satisfying a system of linear inequalities: given a vector $q \in \mathbb{R}^n$ and a matrix $M \in \mathbb{R}^{n \times n}$, find an $x \in \mathbb{R}^n$ such that,

$$\begin{aligned} x &\geq 0 \\ q + Mx &\geq 0 \\ x^T (q + Mx) &= 0 \end{aligned}$$

The LCP

If we let $s = q + Mx$ and $K = \mathbb{R}^n_+ = K^*$, then this can be rewritten as,

$$\left.\begin{array}{ll} x & \in K \\ s & \in K^* \\ \langle x, s \rangle & = 0 \end{array}\right\} (x, s) \in C(K)$$

In other words, the problem can be completely described in terms of the complementarity set $C(K)$ of K.

The LCP

It is noted in [2] that $\mathbb{R}^n_+$ has Lyapunov rank $\beta(\mathbb{R}^n_+) = n$. This is reflected in the fact that the condition $\langle x, s \rangle = 0$ in the linear complementarity problem can be rewritten as n equations,

$$\begin{aligned} x_1 s_1 &= 0 \\ x_2 s_2 &= 0 \\ &\vdots \\ x_n s_n &= 0 \end{aligned}$$

The LCP

Each equation $x_i s_i = 0$ corresponds to an

$$L_i = (\delta_{ii})$$

in $\langle L_i(x), s\rangle = 0$. Since they are obviously linearly-independent, the n transformations $L_1, L_2, \ldots, L_n$ form a basis for $LL(R^n_+)$ and thus $\beta(R^n_+) = n$.

Perfect Cones

Definition (perfect cone).

A proper cone K is said to be *perfect* if $C(K)$ can be expressed in terms of n linearly-independent Lyapunov-like transformations $L_1, L_2, \ldots, L_n$.

That is, if the following two sets are equal:

$$C(K) = \{(x, s) \in K \times K^* : \langle x, s \rangle = 0\}$$

$$\tilde{C}(K) = \bigcap_{i=1}^{n} \{(x, s) \in K \times K^* : \langle L_i x, s \rangle = 0\}$$

Perfect Cones

Example ($K = \mathbb{R}^n_+$). Recall the nonnegative orthant $\mathbb{R}^n_+$ which was used in the linear complementarity problem. We were able to express $C(\mathbb{R}^n_+)$ in terms of n elements of $LL(K)$ equations:

$$\begin{gathered} \langle L_1(x), s\rangle = 0 \\ \vdots \\ \langle L_n(x), s\rangle = 0 \end{gathered}$$

Therefore, $\mathbb{R}^n_+$ is perfect.

Perfect Cones

Let K be a proper cone in $\mathbb{R}^n_+$. Then $1 \implies 2 \implies 3 \implies 4$ [2].

1. $\beta(K) = n$
2. The identity is a linear combination of n independent elements of $LL(K)$.
3. K is perfect
4. $\beta(K) \geq n$

(Clearly, $4 \not\Longrightarrow 1$.)

Perfect Cones

Theorem 1. $2, 3$ and 4 are equivalent.

Proof ($4 \implies 2$).

Suppose $n \leq m = \dim\left(LL\left(K\right)\right)$.

The identity transformation is Lyapunov-like, so we can extend the set $\{I\}$ to a basis $\{I, L_2, \ldots, L_n, \ldots, L_m\}$ of $LL\left(K\right)$.

Now $I = 1I + 0L_2 + \cdots + 0L_n$. □

Bounding the Lyapunov Rank

If our goal is to determine $\beta(K)$ for some K, then it is useful to have an upper bound: if the upper bound is achieved, then $\beta(K)$ is equal to the upper bound.

Clearly, $\dim(LL(K)) \leq \dim(\mathbb{R}^{n \times n}) = n^2$. But we can reduce this bound via the codimension formula:

$$\beta(K) = \operatorname{codim}\left(\operatorname{span}\left\{sx^T : (x, s) \in C(K)\right\}\right)$$

Bounding the Lyapunov Rank

Theorem (Gowda/Tao). For every proper cone K in $\mathbb{R}^n$ with $n \geq 2$, we have $1 \leq \beta(K) \leq n^2 - n$.

Proof. First we note that $\beta(K)$ is invariant under an isomorphism, so for convenience we assume that the standard basis vectors $e_1, e_2, \ldots, e_n$ lie on the boundary of K.

Bounding the Lyapunov Rank

Proof (continued).

The definition of

$$K = (K^*)^* = \{x \in \mathbb{R}^n : \forall y \in K^*,\ \langle x, y \rangle \geq 0\}$$

suggests that every vector e_i on the boundary of K has an associated s_i on the boundary of K^* with $\langle e_i, s_i \rangle = 0$.

Thus, $(e_i, s_i) \in C(K)$. A more technical argument is needed to show that $s_i \neq 0$.

Bounding the Lyapunov Rank

Proof (continued).

If we let $A_i = s_i e_i^T$, then

$$
\begin{aligned}
A_1 &= \left(s_1, 0, \cdots\right) \\
A_2 &= \left(0, s_2, \cdots\right) \\
&\vdots \\
A_n &= \left(0, \cdots, s_n\right)
\end{aligned}
$$

Clearly the A_i are linearly-independent, so $\beta\left(K\right) \leq n^2 - n$ by the codimension formula. $\square$

Bounding the Lyapunov Rank

Theorem 2. For every proper, non-polyhedral cone K in $\mathbb{R}^n$ with $n \geq 3$, we have $1 \leq \beta(K) \leq (n-1)^2$.

The proof of this theorem proceeds in the same way: we use the n matrices constructed by Gowda and Tao, but find an additional $n-1$ pairs $(x, s) \in C(K)$ such that the matrices A_i and sx^T are all linearly independent. This gives us a total of $2n-1$ for our new upper bound of $n^2 - (2n-1) = (n-1)^2$.

Bounding the Lyapunov Rank

Lemma 3. Suppose K is a proper cone in $\mathbb{R}^n$ with $n \geq 2$ whose boundary is contained in a finite union of hyperplanes $\bigcup_{i=1}^{N} H_i$. Then, K is polyhedral.

Proof. The proof is by induction on the number of **non**-supporting-hyperplanes. In the base case, each H_i supports K and thus they define a collection of half-spaces G_i each of which contain K.

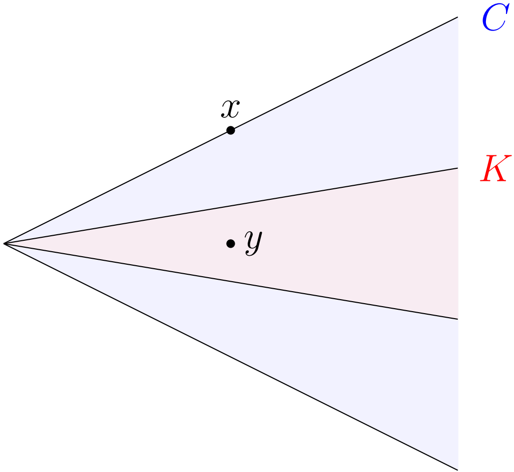
Bounding the Lyapunov Rank

Proof (continued). The cone

$$C = \bigcap_{i=1}^{N} G_i$$

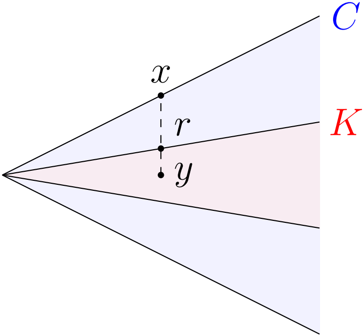
is polyhedral by definition [4] and we claim that $K = C$. Assume that $K \neq C$ on the contrary. Then without loss of generality there exists an $x \in \operatorname{bdy}(C)$ such that $x \notin K$. Now choose some other $y \in \operatorname{int}(K) \subseteq \operatorname{int}(C)$.

Bounding the Lyapunov Rank



Bounding the Lyapunov Rank

Proof (continued). Now we know that the segment $(x, y] \subseteq \operatorname{int}(C)$ and there exists an $r \in (x, y]$ which lies on the boundary of K.



Bounding the Lyapunov Rank

Proof (continued). But by assumption,

$$\operatorname{bdy}(K) \subseteq \bigcup_{i=1}^{N} H_i.$$

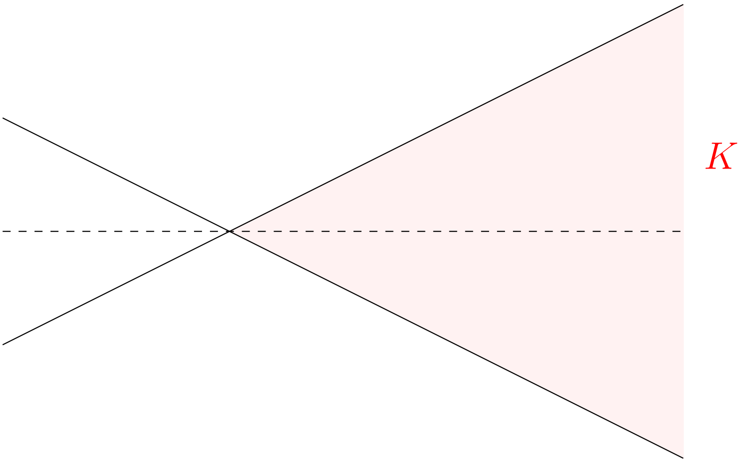
So r must belong to one of the H_i, and $r \in \operatorname{int}(C)$ as well. But each H_i is a supporting hyperplane to C; therefore, $r \in \operatorname{bdy}(C)$, a contradiction.

We conclude that $K = C$ proving that K is polyhedral in the base case.

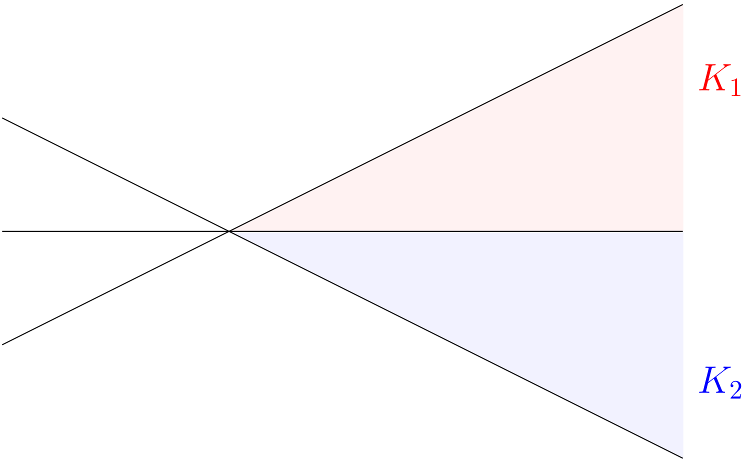
Proof (continued). But what if there are non-supporting hyperplanes (i.e. not the base case)? Any non-supporting-hyperplane must pass through the interior of K splitting it into two smaller cones K_1 and K_2 with $K = K_1 \cup K_2$.

The important observation is that both K_1 and K_2 have one fewer non-supporting-hyperplane than K, allowing us to apply the induction hypothesis.

Bounding the Lyapunov Rank



Bounding the Lyapunov Rank



Proof (continued). At this point we know that K_1 and K_2 are polyhedral and thus finitely-generated:

$$K_1 = \operatorname{conv}\left(\{x_1, x_2, \ldots, x_l\}\right)$$
$$K_2 = \operatorname{conv}\left(\{y_1, y_2, \ldots, y_k\}\right)$$

By convexity of K, we are able to conclude that,

$$K = \operatorname{conv}\left(\{x_1, x_2, \ldots, x_l, y_1, y_2, \ldots, y_k\}\right).$$

Hence, K is polyhedral. $\square$

Bounding the Lyapunov Rank

With Lemma 3 in hand, we are ready to improve the upper bound.

Proof (Theorem 2). We begin with the same $\{A_i\}_{i=1}^{n}$ constructed by Gowda and Tao. Using the same procedure, we can find $0 \neq a_1 \in \operatorname{bdy}(K)$ with $\langle a_1, b_1 \rangle = 0$, i.e. $(a_1, b_1) \in C(K)$. We define a new matrix $B_1 = b_1 a_1^T$, and claim that the set $\{B_1\} \cup \{A_i : i = 1, 2, \ldots, n\}$ is linearly-independent.

Proof (continued). In fact, this procedure can be repeated $(n-1)$ times. Certainly it becomes more difficult with the addition of each successive b_i, so we will assume that we have $(n-2)$ such vectors $b_1, b_2, \ldots, b_{n-2}$, linearly-independent, and find b_{n-1}.

Denote by H_i the set,

$$H_i \coloneqq \operatorname{span}\left\{b_1, b_2, \ldots, b_{n-2}, s_i\right\}$$

Bounding the Lyapunov Rank

Proof (continued). Each H_i defines an $(n-1)$-dimensional space, i.e. a hyperplane. By Lemma 3, we can always find a point on the boundary of K^* not contained in any of the H_i. Take that point to be our $b_{n-1} \in \text{bdy}(K^*)$.

Define $B_i = b_i a_i^T$ as before. We will show that the set $\{B_1, B_2, \dots B_{n-1}, A_1, \dots, A_n\}$ is linearly-independent.

Bounding the Lyapunov Rank

Proof (continued). Let,

$$C_{n-1} = \mu_{n-1} B_{n-1} + \sum_{k=1}^{n-2} \mu_k B_k + \sum_{i=1}^{n} \lambda_i A_i$$

and consider the equation $C_{n-1} = 0$. We will assume that $\mu_{n-1} \neq 0$, and derive a contradiction. The ith column of B_j is $a_j^{(i)} b_j$. Therefore the ith column of C_{n-1} is,

$$\mu_{n-1} a_{n-1}^{(i)} b_{n-1} + \lambda_i s_i + \sum_{k=1}^{n-2} \mu_k a_k^{(i)} b_k$$

Bounding the Lyapunov Rank

Proof (continued). $C_{n-1} = 0$ implies,

$$\mu_{n-1} a_{n-1}^{(i)} b_{n-1} + \lambda_i s_i + \sum_{k=1}^{n-2} \mu_k a_k^{(i)} b_k = 0$$

or,

$$a_{n-1}^{(i)} b_{n-1} = -\frac{\lambda_i}{\mu_{n-1}} s_i - \sum_{k=1}^{n-2} \frac{\mu_k}{\mu_{n-1}} a_k^{(i)} b_k$$
$$\in H_i, \ i = 1, 2, \ldots, n$$

Bounding the Lyapunov Rank

Proof (continued). If $\lambda_i = 0$ and each $\mu_k = 0$, then clearly, $a_{n-1}^{(i)} = 0$ since b_{n-1} is non-zero. On the other hand, if $\lambda_i \neq 0$ or $\mu_k \neq 0$ for some k, then recall that we have chosen $b_{n-1} \notin H_i$, so the only solution to the above equation is $a_{n-1}^{(i)} = 0$.

In both cases, $a_{n-1}^{(i)} = 0$, so we have $a_{n-1}^{(i)} = 0$ for all i, and thus, $a_{n-1} = 0$. But this is a contradiction: we chose a_{n-1} to be non-zero. Therefore the assumption that $\mu_{n-1} \neq 0$ must be at fault.

Proof (continued). But what if $\mu_{n-1} = 0$? Then the equation $C_{n-1} = 0$ reduces to,

$$\lambda_i s_i + \sum_{k=1}^{n-2} \mu_k a_k^{(i)} b_k = 0$$

By assumption, all of the vectors involved are linearly-independent, so all of their coefficients must be zero.

Bounding the Lyapunov Rank

Proof (continued). Adding to these the fact that $\mu_{n-1} = 0$, we have $\lambda_i = \mu_k = 0$, for all k up to $n-1$. Thus we conclude that the set

$$\{B_1, B_2, \ldots, B_{n-1}, A_1, \ldots, A_n\}$$

is linearly-independent, and it contains $2n-1$ elements, giving us an upper bound of $n^2 - (2n-1) = (n-1)^2$. $\square$

Applications

Example ($\beta\left(\mathcal{P}^3_+\right)$). The cone of positive polynomials in $\mathbb{R}^3$ is defined by,

$$\mathcal{P}^3_+ = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : p(t) = x_1 + x_2 t + x_3 t^2 \geq 0\right\}$$

and comprises the coefficient vectors of all nonnegative polynomials $p\left(t\right)$ with $\deg\left(p\right) \leq 2$. It has as its dual the moment cone,

$$\mathcal{M}^3 = \operatorname{conv}\left(\left\{\left(1, t, t^2\right)^T : t \in \mathbb{R}\right\}\right)$$

Example ($\beta\left(\mathcal{P}_{+}^{3}\right)$, continued).

Note that if $x \in \mathcal{P}_{+}^{3}$ and $s \in \mathcal{M}^{3}$, we have $p(t) = \langle x, s \rangle$. In particular, $p(t) \equiv 0 \iff \langle x, s \rangle = 0$. Any such x therefore lies on the boundary of $\mathcal{P}_{+}^{3}$.

Since $p(t) \geq 0$ on all of $\mathbb{R}$, we cannot have $\deg(p) = 1$. Therefore, if $x \neq 0$, we have $\deg(p) = 2$ implying $x_3 \neq 0$. Now the existence of any root implies $x_1 = 0$. Finally, if $p(t)$ has a root, then clearly that root is a double root.

Example ($\beta\left(\mathcal{P}_{+}^{3}\right)$, continued). Consider the following linearly-independent transformations on $\mathbb{R}^3$:

$$L_1 = I \qquad L_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$L_3 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad L_4 = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Applications

Example ($\beta\left(\mathcal{P}^3_+\right)$, continued). To show that each L_i is Lyapunov-like, it suffices ([5], Lemma 25) to exhibit the property for pairs of extreme vectors $(x, s) \in C\left(\mathcal{P}^3_+\right)$.

For our particular problem, we can limit ourselves to a subset of the extreme vectors:

$$\operatorname{Ext}\left(\mathcal{M}^3\right) \ni s \in \left\{\alpha\left(1, t, t^2\right)^T : \alpha > 0, t \in \mathbb{R}\right\}$$

Example ($\beta\left(\mathcal{P}_{+}^{3}\right)$, continued).

The identity is obviously Lyapunov-like, and the other three transformations are easy to check using the fact that p has a double root at t_0:

$$\begin{aligned}
\langle L_2\left(x\right), s\rangle &= x_2 + 2x_3t_0 = p'\left(t_0\right) = 0\\
\langle L_3\left(x\right), s\rangle &= x_2t_0 = 2p\left(t_0\right) - t_0p'\left(t_0\right) = 0\\
\langle L_4\left(x\right), s\rangle &= x_2t_0^2 = 2t_0p\left(t_0\right) - t_0^2p'\left(t_0\right) = 0
\end{aligned}$$

Now from $4 \le \beta\left(\mathcal{P}_{+}^{3}\right) \le 4$ we have $\beta\left(\mathcal{P}_{+}^{3}\right) = 4$.

Applications

Corollary 4. For each $n \geq 3$, there exists a non-symmetric cone $K \subseteq \mathbb{R}^n$ with $\beta(K) > n$.

Proof.

We use the fact that $\mathcal{P}^3_+$ is non-symmetric:

$$K = \mathcal{P}^3_+ \times \mathbb{R}^{n-3}_+ \subseteq \mathbb{R}^3 \times \mathbb{R}^{n-3} \cong \mathbb{R}^n$$
$$K^* = \mathcal{M}^3 \times \mathbb{R}^{n-3}_+ \neq K$$

The Lyapunov rank is additive on a cartesian product, therefore, $\beta(K) = n + 1$. $\square$

What Now?

The previous example shows that the bound $\beta(K) \leq (n-1)^2$ is tight in $n = 3$, since $K = \mathcal{P}_+^3$ achieves the bound of $(3-1)^2 = 4$.

It is not known whether or not the bound is tight for larger n. Perhaps the bound can be improved, or maybe a cone will be found with $\beta(K) = 9$ in $\mathbb{R}^4$. At present, $\beta(\mathcal{L}_+^n) = 7$ is the highest known rank in $\mathbb{R}^4$.

What Now?

For each n, Corollary 4 exhibits a non-symmetric cone K for which $\beta(K) > n$. However, by construction, K is reducible. It is not known whether or not there exist irreducible cones having the same property.

What Now?

What does it mean for one system $\langle L_i x, s \rangle = 0$ to be simpler than another? How can we find those systems?

For linear systems, n equations in n variables is naturally desirable. But a priori, $\langle Lx, s \rangle = 0$ is not linear. Even for simple(?) cones and simple(?) choices of the L_i, the resulting systems can be hard to solve. Are there choices of L_i that make the system easily solvable? Are there cones where no choice of L_i gives us an easy system?

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