Ornstein-Uhlenbeck Processes

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Goal. To introduce a new financial dervative.

- No fun.
- I'm bad at following directions.
- The derivatives based on Geometric Brownian Motion don't model reality anyway.

So we're going to replace the underlying stochastic process instead.

Claim. GBM doesn't work in real life.

Proof.
Bekaert and Hodrick (1992), Bessembinder and Chan (1992), Breen, Glosten, and Jagannathan (1989), Campbell and Ammer (1993), Campbell and Hamao (1992), Chan (1992), Chen (1991), Chen, Roll, and Ross (1986), Chopra, Lakonishok, and Ritter (1992), DeBondt and Thaler (1985), Engle, Lilien, and Robbins (1987), Fama and French (1988a, 1988b, 1990), Ferson (1989, 1990), Ferson, Foerster, and Keim (1993), Ferson and Harvey (1991a, 1991b), Ferson, Kandel, and Stambaugh (1987), Gibbons and Ferson (1985), Harvey (1989), Jegadeesh (1990), Keim and Stambaugh (1986), King (1966), Lehmann (1990), Lo and MacKinlay (1988, 1990, 1992), and Poterba and Summers (1988). □

Wiener Processes

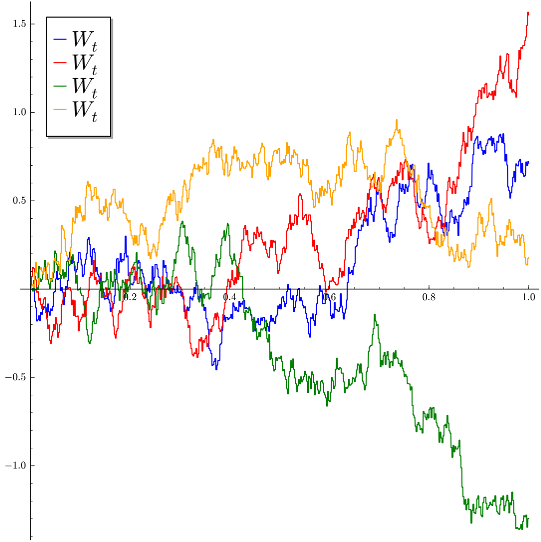
Definition (Wiener Process). You should know this already [1] (Chapter 4).

Recall two important properties:

1. $W_0 = 0$.
2. $s < t$ implies $[W_t - W_s] \sim N\left[0, \sqrt{t-s}\right]$

Note. As the time period $t - s$ approaches infinity, so does the standard deviation!

Wiener Processes



Wiener Processes

What's wrong here?

- If you're unlucky, the Wiener process can go bonkers.
- When it does, it tends to stay there.

This doesn't accurately reflect what happens in real life.

Geometric Brownian Motion

The Black-Scholes model that we've been using assumes that the stock dynamics follow Geometric Brownian Motion:

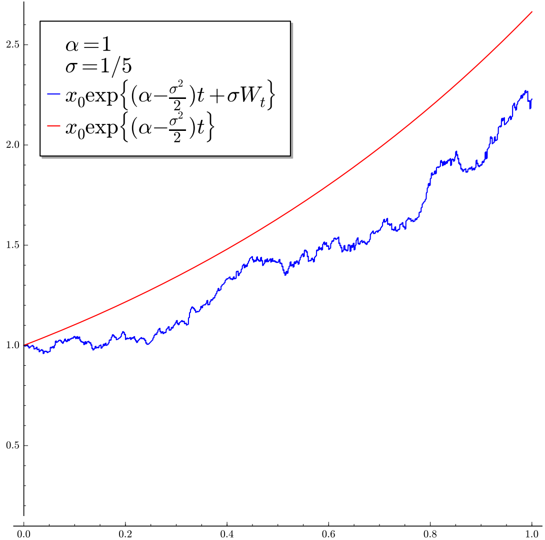
$$\begin{aligned} dX_t &= \alpha X_t dt + \sigma X_t dW_t \\ X_0 &= x_0 \end{aligned}$$

with solution,

$$X_t = x_0 \exp\left\{ \left(\alpha - \frac{\sigma^2}{2} \right) t + \sigma W_t \right\}$$

Since this contains W_t, it inherits all of its problems.

Geometric Brownian Motion



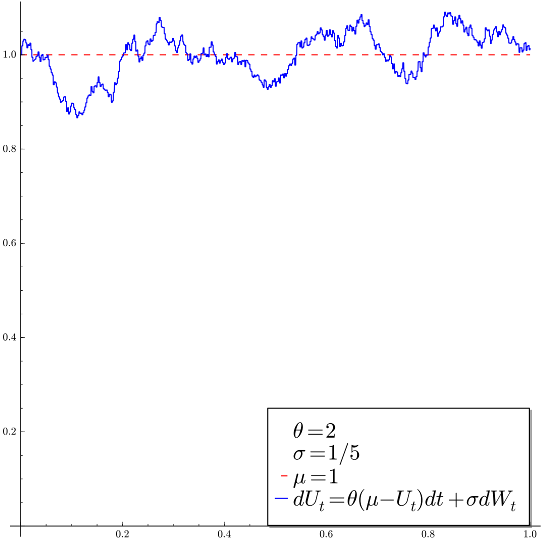
Definition (Ornstein-Uhlenbeck Process). The Ornstein-Uhlenbeck process is a stochastic process with dynamics,

$$
\begin{aligned}
dU_t &= \theta \left(\mu_t - U_t \right) dt + \sigma dW_t \\
U_0 &= u_0
\end{aligned}
$$

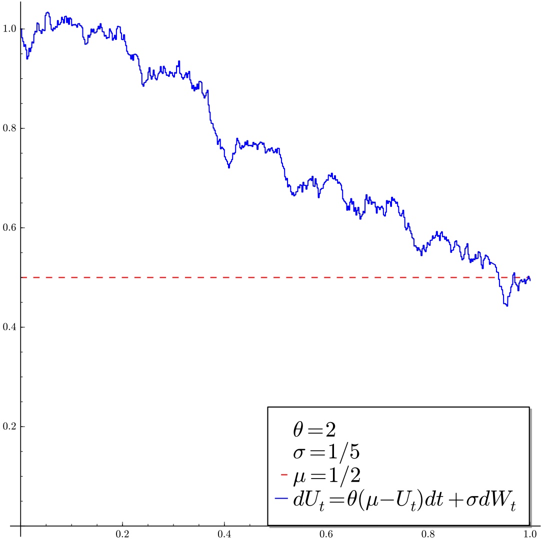
where W_t is a Wiener process.

- Can be seen as a modification of a Wiener process.
- μ_t is the mean of the process.
- θ is the tendency of the process to return to the mean.

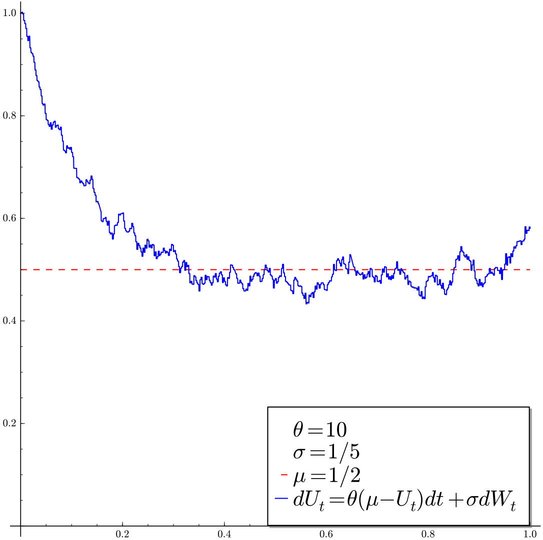
Ornstein-Uhlenbeck



Ornstein-Uhlenbeck



Ornstein-Uhlenbeck



Ornstein-Uhlenbeck

Theorem. When μ is a constant, the solution to the Ornstein-Uhlenbeck SDE is given by,

$$U_t = \mu \left[1 - e^{-\theta t} \right] + x_0 e^{-\theta t} + \sigma N \left[0, \xi \right]$$

where,

$$\xi = \sqrt{\frac{1}{2\theta} \left[1 - e^{-2\theta t} \right]}$$

Proof.

- Look up the solution on Wikipedia.
- Assume that it's correct.
- Then, the solution is of the form,

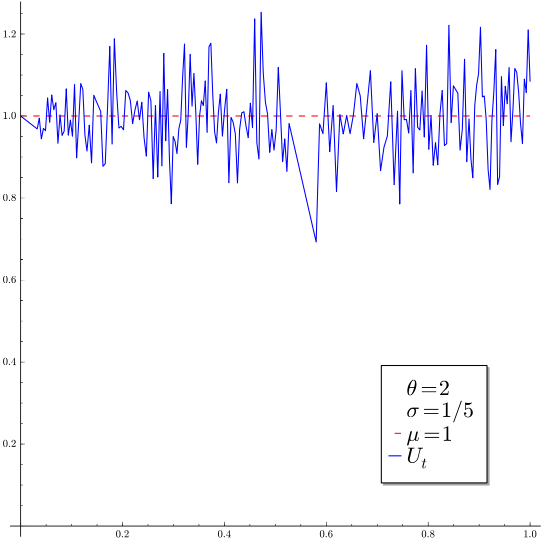
$$f(x_t, t) = x_t e^{\theta t}$$

- Apply Itô's formula.

$\square$

(or you can apply Proposition 5.3 from our textbook with $A = -\theta$ and $b_t = \mu\theta$)

Ornstein-Uhlenbeck



Lo & Wang Process

That's great, but we want to price some derivatives.

Let P_t be some price process, and $p_t = \ln(P_t)$. We're going to assume that p_t has the dynamics,

$$dp_t = \left(-\theta\left(p_t - \eta t\right) + \eta\right) dt + \sigma dW_t$$
$$p_0 = p_0$$

We see that p_t is shifted by ηt – we want to get rid of that.

Lo & Wang Process

We can rewrite that SDE as,

$$d\left(p_t - \eta t\right) = -\theta\left(p_t - \eta t\right)dt + \sigma dW_t$$

Now if we let $\eta t = \mu_t$, the right-hand side looks like the Ornstein-Uhlenbeck SDE!

$$d\left(p_t - \mu_t\right) = -\theta\left(p_t - \mu_t\right)dt + \sigma dW_t$$

Lo & Wang Process

We'll rename $p_t - \mu_t = q_t$ to reduce the amount of ugly. Thus we have,

$$dq_t = -\theta q_t dt + \sigma dW_t$$

Compare with Geometric Brownian Motion:

$$d\ln\left(X_t\right) = \left(\alpha - \frac{\sigma^2}{2}\right)dt + \sigma dW_t$$

Since we have a function of t in the dt term, our process is actually a slight generalization of GBM. It allows us to model trends and predictability.

Lo & Wang Process

Oh, and we can still solve our new process explicitly [3]:

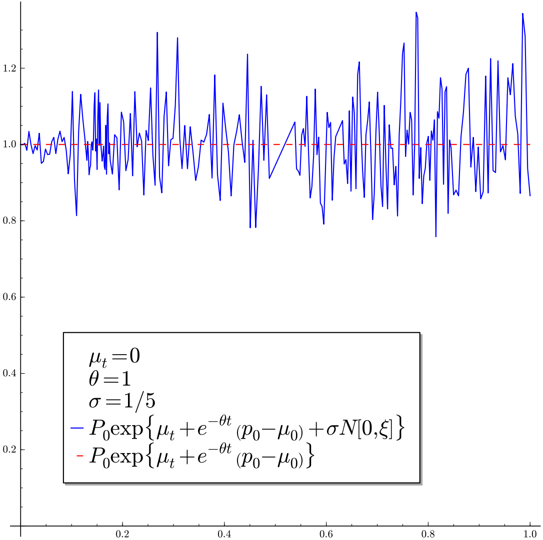
$$q_t = e^{-\theta t} q_0 + \sigma \int_0^t e^{-\theta(t-s)} dW_s$$

Translating back into the original price process...

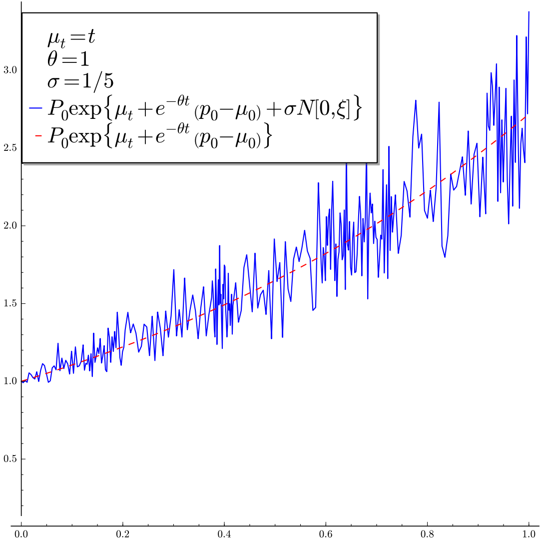
$$p_t = \mu_t + e^{-\theta t} \left(p_0 - \mu_0\right) + \sigma N \left[0, \xi\right]$$

where again,

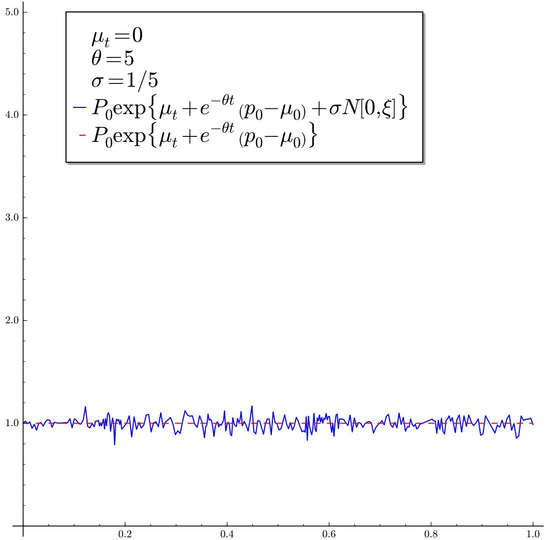
$$\xi = \sqrt{\frac{1}{2\theta} \left[1 - e^{-2\theta t}\right]}$$


1.2
1.0
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8
1.0
$\mu_t = 0$
$\theta = 1$
$\sigma = 1/5$
$P_0 \exp\left\{\mu_t + e^{-\theta t}(p_0 - \mu_0) + \sigma N[0,\xi]\right\}$
$P_0 \exp\left\{\mu_t + e^{-\theta t}(p_0 - \mu_0)\right\}$

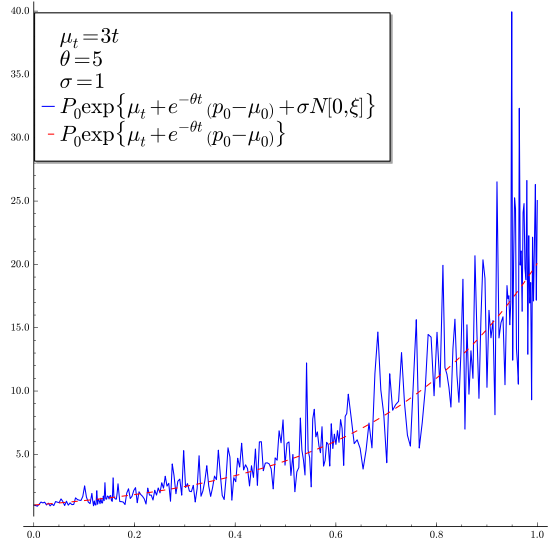
Lo & Wang Process



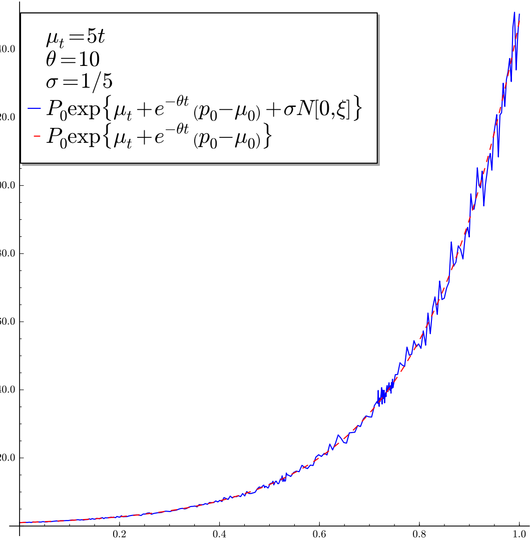
Lo & Wang Process



Lo & Wang Process



Lo & Wang Process



Theorem. The Black-Scholes formula works for the process we defined earlier, q_t.

This sounds rather amazing at first (see [3] for a citation), but there is a caveat. While the *formula* still works, the numerical value of σ will differ between GBM and Ornstein-Uhlenbeck.

The short version:

$$\sigma^2_{OU} = \sigma^2_{GBM} \cdot \left[\theta\tau\left(1 - e^{-\theta\tau}\right)^{-1}\right]$$

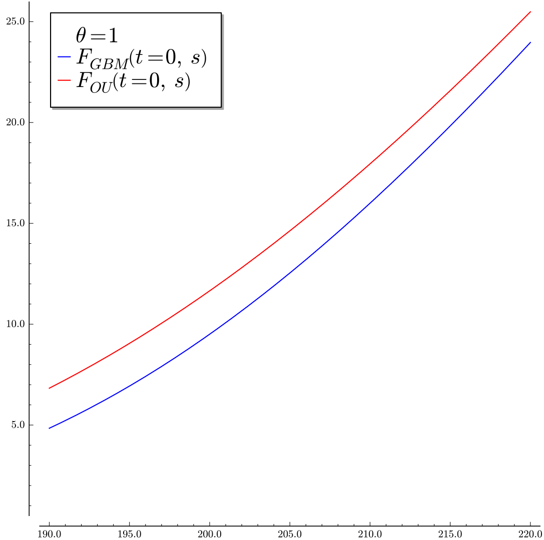
(Here, τ is the length of the time period over which your observations are made.)

Example (IBM, March 2012).

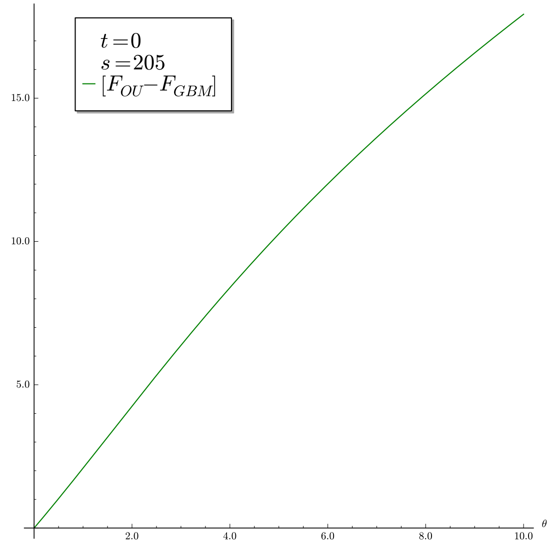
Table: Observed prices for March 2012

Date	*Closing Price*
2012-03-01	whatever
2012-03-02	whatever
2012-03-05	whatever
2012-03-06	whatever
...	...
2012-03-30	whatever

Option Pricing



Option Pricing



As θ increases, the tendency to vary from the mean decreases. Therefore, a large θ corresponds to a "predictable" stock.

Conclusion. Options on a predictable stock are more valuable!

References

[1] Björk, Tomas. Arbitrage Theory in Continuous Time, 3rd Ed. Oxford University Press, Oxford, 2009.

[2] Doob, J. L. The Brownian Movement and Stochastic Equations. The Annals of Mathematics, Second Series, Vol. 43, No. 2 (Apr., 1942), pp. 351-369.

[3] Lo, A. W.; Wang, J.: Implementing Option Pricing Models when asset returns are predictable. The Journal of Finance 50, pp. 87–129, 1995.

[4] Stein, William A. et al. Sage Mathematics Software (Version 5.0), The Sage Development Team, 2012, http://www.sagemath.org.

[5] Uhlenbeck, G. E. and Ornstein, L. S. On the Theory of Brownian Motion. Phys. Rev. 36, pp. 823-841. 1930.