The Lyapunov rank of an improper cone

Part II - Lie algebra

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Recap

In Part I, we generalized existing results for proper cones to the class of closed convex cones.

Closed convex cones are specified in terms of *generators* rather than extreme vectors (which a closed convex cone may not have).

Recap

To perform operations in subspaces, we tag a cone K with its ambient space V and write K_V.

The result is called a cone-space pair. All operations on closed convex cones can be defined on cone-space pairs in an obvious way.

Recap

One cone-space pair K_S will appear frequently. We define

$$S := \operatorname{span}(K_V)$$

so that K_S is "K restricted to its own span."

Recap

If G_1 generates K_V and G_2 generates its dual, then the Lyapunov-like property,

$$\langle L(x), s \rangle = 0 \text{ for all } (x, s) \in C(K_V)$$

need only be checked on $x \in G_1$ and $s \in G_2$.

Recap

The codimension formula describes the space of all Lyapunov-like transformations on K_V:

$$\mathbf{LL}\left(K_V\right) = \operatorname{span}\left(\left\{ s \otimes x \mid (x, s) \in C\left(K_V\right) \right\}\right)^{\perp}.$$

We need only consider generators above, thus we can compute $\mathbf{LL}\left(K_V\right)$ when K_V is polyhedral.

Recap

The pointed/solid duality reduces the Lyapunov rank computation to that of a proper cone:

Theorem.

$$\begin{aligned}\beta\left(K_V\right) = {} & \beta\left(K_{SP}\right) \\ & + \operatorname{lin}\left(K\right) \cdot \dim\left(K\right) \\ & + \operatorname{codim}\left(K_V\right) \cdot \dim\left(V\right).\end{aligned}$$

Recap

The proof of this theorem uses the fact that K_V is isomorphic to $K_S \times \{0\}_{S^\perp}$. And,

$$\mathbf{LL}\left(K_S \times \{0\}_{S^\perp}\right) = \left\{ \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \right\}$$

$$\begin{aligned} A &\in \mathbf{LL}\left(K_S\right) \\ B &= \text{whatever} \\ D &= \text{whatever}. \end{aligned}$$

The problem

An interesting connection [2] for proper cones:

Theorem (Gowda/Tao).

The following are equivalent when K_V is proper.

- L is Lyapunov-like on K_V.
- $e^{tL} \in \operatorname{Aut}(K_V)$ for all $t \in \mathbb{R}$.
- $L \in \operatorname{Lie}(\operatorname{Aut}(K_V))$.

The problem

So what do we need to prove for cone-space pairs?

$$L \text{ is Lyapunov-like on } K_V$$
$$\Updownarrow$$
$$e^{tL} \in \operatorname{Aut}(K_V) \text{ for all } t \in \mathbb{R}.$$

One implication requires no modification.

The problem

Proposition.

Let K_V be a cone-space pair. If $e^{tL} \in \operatorname{Aut}(K_V)$ for all $t \in \mathbb{R}$, then L is Lyapunov-like on K_V.

Proof.

Let $e^{tL} \in \operatorname{Aut}(K_V)$ for all $t \in \mathbb{R}$. We will see that $\langle L(x), s \rangle = 0$ for any $(x, s) \in C(K_V)$.

The problem

Proof (continued). Since $e^{tL}(x) \in K$,

$$\left\langle e^{tL}(x), s \right\rangle = \left\langle \left[e^{tL} - I \right](x), s \right\rangle \geq 0.$$

We can divide by $1/t > 0$:

$$\left\langle \underbrace{\frac{1}{t} \left[e^{tL} - I \right]}_{\text{looks derivativy}} (x), s \right\rangle \geq 0 \text{ for all } t > 0.$$

The problem

Proof (continued). Take the limit to turn this into a derivative evaluation,

$$L = \lim_{t \to 0} \left\{ \frac{1}{t} \left[e^{tL} - I \right] \right\} = \frac{d}{dt} e^{tL} \bigg|_{t=0}$$

Thus $\langle L(x), s \rangle \geq 0$.

The same trick with $-L$ gives $\langle L(x), s \rangle \leq 0$. $\square$

The problem

We still need implication in the other direction:

$$L \in \mathbf{LL}\left(K_V\right) \implies e^{tL} \in \operatorname{Aut}\left(K_V\right).$$

How might we prove something like this for closed convex cones?

Proper cones

The proof for proper cones centers on a theorem [5] of Schneider and Vidyasagar.

Theorem.

Let K be a proper cone in $\mathbb{R}^n$ and let A be a matrix in $\mathbb{R}^{n \times n}$. Then $A \in \Sigma\left(K\right)$ if and only if $e^{tA}\left(K\right) \subseteq K$ for all $t \geq 0$.

Proper cones

Note that $\Sigma(K)$ is nothing but $-\mathbf{Z}(K)$. If we stick a negative sign in front of A, we get:

Theorem.

Let K be a proper cone in $\mathbb{R}^n$ and let A be a matrix in $\mathbb{R}^{n \times n}$. Then $-A \in \Sigma(K)$ if and only if $e^{tA}(K) \subseteq K$ for all $t \leq 0$.

Proper cones

Now if both A and $-A$ are in $\Sigma(K)$, then they're both in $\mathbf{Z}(K)$, and thus $A \in \mathbf{LL}(K)$. So,

$$A \in \mathbf{LL}(K)$$

$$\Updownarrow$$

$$e^{tA}(K) \subseteq K \text{ for all } t \in \mathbb{R}.$$

Proper cones

We recognize that $\left(e^{tA}\right)^{-1} = e^{-tA}$, and clearly,

$$e^{-tA}(K) \subseteq K \text{ for all } t \in \mathbb{R}$$
$$\Updownarrow$$
$$e^{tA}(K) \subseteq K \text{ for all } t \in \mathbb{R}.$$

Combining the two, $e^{tA} \in \operatorname{Aut}(K)$ for all $t \in \mathbb{R}$.

This is equivalent [1] to $A \in \operatorname{Lie}(\operatorname{Aut}(K))$.

Proper cones

As a result, we have the Gowda/Tao equivalence:

- L is Lyapunov-like on K_V.
- $e^{tL} \in \operatorname{Aut}(K_V)$ for all $t \in \mathbb{R}$.
- $L \in \operatorname{Lie}(\operatorname{Aut}(K_V))$.

Proper cones

If Schneider's result extends to closed convex cones, the Lie algebra connection will too.

Schneider's proof begins, "Let $A \in \Sigma(C)$. Then by Theorem 2 and Lemma 6…"

So we work backwards.

Proper cones

Lemma 6 states:

> *Let C be a proper cone in $\mathbb{R}^n$. Then in $\mathbb{R}^{n \times n}$, the closure of $\Sigma^+(C)$ is $\Sigma(C)$.*

So what's $\Sigma^+(C)$? It's the set of all *strictly cross-positive* matrices on C, of course.

Proper cones

Definition.

Let C be a cone in $\mathbb{R}^n$. An $A \in \mathbb{R}^{n \times n}$ is called *strictly cross-positive* on C if for all $0 \neq y \in C$, $0 \neq z \in C^*$ with $y \perp z$ we have $\langle Ay, z \rangle > 0$.

Uh oh. Since both y and $-y$ belong to a subspace, Lemma 6 cannot hold for subspaces.

Automorphisms

What *can* we do (grasping at straws)?

If we work with the convenient form of a non-solid cone, we can actually compute its automorphism group. Suppose

$$L := \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \operatorname{Aut}\left(K_S \times \{0\}_{S^\perp}\right).$$

Well, $C \equiv 0$, otherwise there's some $x \in K_S$ with $C(x) \neq 0$ and thus,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} A(x) \\ C(x) \end{bmatrix} \notin \begin{bmatrix} K_S \\ \{0\}_{S^\perp} \end{bmatrix}$$

Automorphisms

Now B and D can be *almost* anything, since,

$$\begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \begin{bmatrix} K_S \\ 0 \end{bmatrix} = \begin{bmatrix} A\left(K_S\right) \\ 0 \end{bmatrix}$$

But we want L^{-1} to exist, so D^{-1} must.

And we see that we will need $A \in \operatorname{Aut}\left(K_S\right)$.

Automorphisms

This is also a sufficient condition for membership in $\operatorname{Aut}\left(K_S \times \{0\}_{S^\perp}\right)$. Thus,

$$\operatorname{Aut}\left(K_S \times \{0\}_{S^\perp}\right) = \left\{ \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \right\}$$

$$A \in \operatorname{Aut}\left(K_S\right)$$

$$B = \text{whatever}$$

$$D \in \operatorname{Aut}\left(S^\perp\right).$$

Automorphisms

This looks a lot like our description of

$$\mathbf{LL}\left(K_S \times \{0\}_{S^\perp}\right) = \left\{ \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \right\}$$

$$A \in \mathbf{LL}\left(K_S\right)$$

$$B = \text{whatever}$$

$$D = \text{whatever}\ldots$$

but it's not clear how that helps.

Automorphisms

Suppose we could get the result for $K_S \times \{0\}_{S^\perp}$. Could we extend it to K_V?

Proposition. Let $K_V \cong J_W$ be isomorphic cone-space pairs with $K_V = \psi\left(J_W\right)$. Then,

- $\operatorname{Aut}\left(J_W\right) = \psi \operatorname{Aut}\left(K_V\right)\psi^{-1}$.
- $\mathbf{LL}\left(J_W\right) = \psi \mathbf{LL}\left(K_V\right)\psi^{-1}$.
- $e^{\psi L \psi^{-1}} = \psi e^{L} \psi^{-1}$.

Automorphisms

Proof. The first two proofs are similar.

$$\begin{aligned}
& L \in \mathbf{LL}\left(K_V\right) \\
\iff & \left\langle L\left(x\right), s\right\rangle = 0 \text{ for all } \left(x, s\right) \in C\left(K_V\right) \\
\iff & \left\langle L\left(\psi^{-1} y\right), \psi^{-1} t\right\rangle = 0 \text{ for all } \left(y, t\right) \in C\left(J_W\right) \\
\iff & \left\langle \psi L \psi^{-1}\left(y\right), t\right\rangle = 0 \text{ for all } \left(y, t\right) \in C\left(J_W\right) \\
\iff & \psi L \psi^{-1} \in \mathbf{LL}\left(J_W\right)
\end{aligned}$$

Automorphisms

Proof (continued).

For the exponential, we realize that $e^{\psi L \psi^{-1}}$ is a sum of powers of $\psi L \psi^{-1}$. But,

$$\left(\psi L \psi^{-1}\right)^n = \underbrace{\psi L \overbrace{\psi^{-1} \psi}^{I} L \psi^{-1} \cdots \psi L \psi^{-1}}_{n \text{ times}},$$

so the ψ all cancel except at the ends. $\square$

Automorphisms

Now a result for $K_S \times \{0\}_{S^\perp}$ becomes one for K_V.

Proposition.

$$\mathbf{LL}\left(K_S \times \{0\}_{S^\perp}\right) = \operatorname{Lie}\left(\operatorname{Aut}\left(K_S \times \{0\}_{S^\perp}\right)\right)$$
$$\Updownarrow$$
$$\mathbf{LL}\left(K_V\right) = \operatorname{Lie}\left(\operatorname{Aut}\left(K_V\right)\right)$$

Automorphisms

Proof.

Let $\phi\left(K_V\right) = K_S \times \{0\}_{S^\perp}$ and $L \in \mathbf{LL}\left(K_V\right)$. Then,

$$\phi L \phi^{-1} \in \mathbf{LL}\left(K_S \times \{0\}_{S^\perp}\right)$$
$$\Updownarrow$$
$$e^{t\phi L \phi^{-1}} = \phi e^{tL} \phi^{-1} \in \operatorname{Aut}\left(K_S \times \{0\}_{S^\perp}\right)$$

Proof (continued).

But we can rearrange the isomorphisms:

$$
\begin{array}{ccc}
L \in \phi^{-1}\mathbf{LL}\left(K_S \times \{0\}_{S^\perp}\right)\phi & = & \mathbf{LL}\left(K_V\right) \\
\Updownarrow & & \Updownarrow \\
e^{tL} \in \phi^{-1}\operatorname{Aut}\left(K_S \times \{0\}_{S^\perp}\right)\phi & = & \operatorname{Aut}\left(K_V\right)
\end{array}
$$

$\square$

Automorphisms

Where we are: if we can show that

$$L \in \mathbf{LL}\left(K_S \times \{0\}_{S^\perp}\right)$$
$$\Longrightarrow$$
$$e^{tL} \in \operatorname{Aut}\left(K_S \times \{0\}_{S^\perp}\right) \text{ for all } t \in \mathbb{R},$$

then we can use the last proposition to show the same thing for any K_V.

Automorphisms

Great. Now all we need is some high-powered mathematics to give us the result for $K_S \times \{0\}_{S^\perp}$.

Wait a minute, what was $\mathbf{LL}\left(K_S \times \{0\}_{S^\perp}\right)$?

$$L \coloneqq \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$$

$$A \in \mathbf{LL}\left(K_S\right)$$

$$B = \text{whatever}$$

$$D = \text{whatever.}$$

Automorphisms

What if we just…exponentiate this?

$$e^{tL} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}^k.$$

Automorphisms

The powers of an upper-triangular matrix are upper-triangular!

$$\begin{aligned} e^{tL} &= \begin{bmatrix} \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k & \widetilde{B} \\ 0 & \sum_{k=0}^{\infty} \frac{t^k}{k!} D^k \end{bmatrix} \\ &= \begin{bmatrix} e^{tA} & \widetilde{B} \\ 0 & e^{tD} \end{bmatrix}. \end{aligned}$$

Ha ha!

Automorphisms

K_S might not be proper, but if it is, this matches our description of $\operatorname{Aut}\left(K_S \times \{0\}_{S^\perp}\right)$:

$$\operatorname{Aut}\left(K_S \times \{0\}_{S^\perp}\right) = \left\{ \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \right\}$$

$$A \in \operatorname{Aut}\left(K_S\right)$$

$$B = \text{whatever}$$

$$D \in \operatorname{Aut}\left(S^\perp\right).$$

AUTOMORPHISMS

Why?

- If K_S is proper, then $e^{tA} \in \operatorname{Aut}(K_S)$ by the Gowda/Tao result.
- The exponential e^{tD} is always invertible.
- Whatever $\widetilde{B}$ is, it doesn't matter.

Automorphisms

How can we make K_S proper?

Lemma. Suppose K_V is pointed. Then the following are equivalent.

- L is Lyapunov-like on K_V.
- $e^{tL} \in \operatorname{Aut}(K_V)$ for all $t \in \mathbb{R}$.
- $L \in \operatorname{Lie}(\operatorname{Aut}(K_V))$.

Proof. Just exponentiate. $\square$

Automorphisms

Can we get rid of the pointed requirement?

Proposition. For any cone-space pair K_V,

- $\operatorname{Aut}(K_V^*) = \{A^* \mid A \in \operatorname{Aut}(K_V)\}$.
- $\mathbf{LL}(K_V^*) = \{L^* \mid L \in \mathbf{LL}(K_V)\}$.
- $e^{t(L^*)} = \left(e^{tL}\right)^*$.

(These proofs are trivial.)

Automorphisms

We can reuse our trick from Part I.

Lemma. Suppose K_V is solid. Then the following are equivalent.

- L is Lyapunov-like on K_V.
- $e^{tL} \in \operatorname{Aut}(K_V)$ for all $t \in \mathbb{R}$.
- $L \in \operatorname{Lie}(\operatorname{Aut}(K_V))$.

Automorphisms

Proof.

K_V^* is pointed, so

$$\begin{array}{ccc} L^* \in \mathbf{LL}\left(K_V^*\right) & \iff & L \in \mathbf{LL}\left(K_V\right) \\ \Updownarrow & & \Updownarrow \\ e^{tL^*} \in \operatorname{Aut}\left(K_V^*\right) & \iff & e^{tL} \in \operatorname{Aut}\left(K_V\right) \end{array}$$

□

Automorphisms

Where we are:

We started by assuming that K_V was pointed…so that K_S would be proper…so that we could apply the Gowda/Tao theorem.

But we just found a version of the Gowda/Tao theorem that requires only a solid cone.

Automorphisms

Theorem. Let K_V be a cone-space pair. The following are equivalent.

- L is Lyapunov-like on K_V.
- $e^{tL} \in \operatorname{Aut}(K_V)$ for all $t \in \mathbb{R}$.
- $L \in \operatorname{Lie}(\operatorname{Aut}(K_V))$.

Proof. Go back and prove everything without pointedness (K_S is already solid). $\square$

Automorphisms

Example. Let $K = V = \mathbb{R}^n$.

Then $\operatorname{Aut}(K_V) = \mathbf{GL}_n(\mathbb{R})$, and we know [1] that,

$$\operatorname{Lie}(\mathbf{GL}_n(\mathbb{R})) = \mathbf{M}_n(\mathbb{R}),$$

the set of $n \times n$ real matrices. And of course,

$$\mathbf{GL}_n(\mathbb{R}) = \exp(\mathbf{M}_n(\mathbb{R})).$$

Miscellaneous results

Theorem.

Suppose K_V is a polyhedral cone-space pair with finite generating set G.

1. If every element of G is an eigenvector of L, then $L \in \mathbf{LL}\left(K_V\right)$.
2. If $L \in \mathbf{LL}\left(K_V\right)$, then every extreme vector of K_V is an eigenvector of L.

Miscellaneous results

Proof.

Suppose every $x \in G$ is an eigenvector of L. To verify the Lyapunov-like property on generators,

$$\langle L(x), s \rangle = \langle \alpha x, s \rangle = \alpha \langle x, s \rangle = 0,$$

because $(x, s) \in C(K_V)$.

Miscellaneous results

Proof (continued).

Suppose $L \in \mathbf{LL}(K_V)$. Gowda and Tao use $e^{tL} \in \operatorname{Aut}(K_V)$ to prove that every $\operatorname{Ext}(K_V)$ is an eigenvector of L when K_V is proper.

The same argument works using the new result that $e^{tL} \in \operatorname{Aut}(K_V)$ for cone-space pairs.

Miscellaneous results

Everything should be made as simple as possible, but not simpler.

— Albert Einstein

(maybe)

Miscellaneous results

Theorem (incorrect).

Suppose K_V is a polyhedral cone-space pair. Then $L \in \mathbf{LL}(K_V)$ if and only if every element of $\operatorname{Ext}(K_V)$ is an eigenvector of L.

Miscellaneous results

Looks great, but is too simple.

Suppose every element of $\operatorname{Ext}(K_V)$ is an eigenvector of L. Do we have $L \in \mathbf{LL}(K_V)$?

- Works for proper cones.
- Works for $K = V$.
- So it's got to be true.

Example (counter). Let $K = \mathbb{R}^2$ in $V = \mathbb{R}^3$.

Clearly $\operatorname{Ext}(K_V) = \emptyset$, yet not every $L \in \mathcal{B}(V)$ is Lyapunov-like on K_V: we know that $\beta(K_V) = 7$.

Oops.

Future work

Our Lie algebra theorem is weaker than the result of Schneider and Vidyasagar stating that $L \in \Sigma(K)$ if and only if $e^{tL}(K) \subseteq K$.

The maps P satisfying $P(K) \subseteq K$ are called *positive maps* on K, and the set of all such maps is written $\pi(K)$.

Future work

The techniques we've developed can be used to generalize their theorem to one concerning $\mathbf{Z}(K_V)$ and $\pi(K_V)$ for a cone-space pair K_V.

When K_V is polyhedral, the set $\pi(K_V)$ is a polyhedral closed convex cone, and we should be able to compute it.

Future work

The positive maps are interesting in their own right, and having an implementation may generate some conjectures about e.g. $\beta(\pi(K_V))$.

Moreover, $\mathbf{Z}$-transformations arise in dynamical systems, so we should be able to connect this work to that area and extend some of the results there to closed convex cones.

References I

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[3] J. Hilgert, K. Hofmann, and J. Lawson. *Lie groups, convex cones and semigroups*, Oxford University Press, Oxford, 1989.

References II

[4] M. Orlitzky. The Lyapunov rank of an improper cone (preprint).

[5] H. Schneider and M. Vidyasagar, *Cross-positive matrices*, SIAM Journal on Numerical Analysis, 7 (1970) 508–519.