# On **Z**-operators and viability theorems

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**Abstract**

Let $K$ be a closed convex cone with dual $K^*$ in a finite-dimensional real Hilbert space $V$. If $L$ is a linear operator on $V$, then one definition says that $L$ is a **Z**-operator on $K$ if

$$\langle L(x), s \rangle \leq 0 \text{ for all } (x, s) \in K \times K^* \text{ such that } \langle x, s \rangle = 0.$$

The **Z**-operators generalize matrices whose off-diagonal elements are nonpositive, and they arise in many applications. It is known that $-L$ is exponentially-positive on $K$ if and only if $L$ is a **Z**-operator on $K$. The outward normal cone to $K$ at a point $x$ on its boundary is

$$N_K(x) := \{ v \in V \mid \langle v, y - x \rangle \leq 0 \text{ for all } y \in K \},$$

and we say that $z \in V$ is *subtangential to* $K$ *at* $x$ if $-z \in N_K(x)^*$. It is also known that $-L$ is exponentially-positive on $K$ if and only if $L(x)$ is subtangential to $K$ at every point $x$ on its boundary. The concept of subtangentiality is thus apparently connected to that of a **Z**-operator. We show that the connection can be be made explicit, providing an intuitive geometric interpretation for our definition of a **Z**-operator. This connects the **Z**-operators to several viability theorems in dynamical systems and elucidates a connection with Lie theory.

## 1 Introduction

A **Z**-matrix is a real square matrix whose off-diagonal entries are nonpositive. The **Z**-matrices are an important class of matrices in optimization; for example, there are existence results for linear complementarity problems involving a **Z**-matrix [6]. The **Z**-matrices thus have applications in game theory and linear programming, as well as the other fields that fall under the umbrella of complementarity. Any matrix of the form $\lambda I - N$ where $\lambda \in \mathbb{R}$ and $N$ has nonnegative entries is a **Z**-matrix. When $\lambda$ dominates the spectral radius of $N$, we call $\lambda I - N$ an **M**-matrix. The **M**-matrices are thus a subclass of the **Z**-matrices. Berman and Plemmons [2] devote an entire chapter to nonsingular **M**-matrices, connecting **Z**-matrices to many more areas.

The main question we are concerned with is how to generalize a **Z**- or **M**-matrix to cones other than $\mathbb{R}^n_+$. The existence results for complementarity problems are predicated on the fact that the feasible region for a complementarity problem involving a **Z**-matrix forms a meet semi-sublattice that will have a "least" element. One idea is to extend the properties of a **Z**-matrix from $\mathbb{R}^n$ to a more general vector lattice [4, 7]. While this approach is historically relevant (and has merits that our approach does not), we don't pursue it further here.

Going forward, we adopt a generalization due collectively to Schneider and Vidyasagar [23], Stern and Tsatsomeros [25, 26], and Gowda and Tao [11]: if $K^*$ represents the dual of a closed convex cone $K$, then $L$ is a **Z**-operator on $K$ and we write $L \in \mathbf{Z}(K)$ if $\langle L(x), s\rangle \leq 0$ for all $x \in K$ and $s \in K^*$ such that $\langle x, s\rangle = 0$. Schneider and Vidyasagar worked with this same set of operators, defined on a *proper* cone and modulo a negative sign, in the context of the exponential-positivity that characterizes them. Later, Gowda and Tao considered them as explicit generalizations of **Z**-matrices (again on *proper* cones), and that definition was extended directly to closed convex cones. In the meantime Stern and Tsatsomeros worked with the concept of subtangentiality that, in the right setting, is equivalent to exponential-positivity and thus the property that defines a **Z**-operator. Simultaneously, Hilgert and Hofmann [13] were studying invariant wedges in relation to Lie theory and derived many of the same results.

Ultimately, our goal is to show that the recent definition of and results for **Z**-operators on closed convex cones can be recovered from the subtangentiality condition used by Stern and Tsatsomeros [25, 26]. This has the benefit of being immediately applicable to non-proper closed convex cones, and provides an intuitive geometric reason for why **Z**-operators on closed convex cones are exponentially-positive. The concept of subtangentiality can be traced back to several viability theorems in dynamical systems, so-named because they govern whether or not the evolution of a system is viable (satisfies some constraints) at any given time. In our case, the constraint is that the trajectory remains inside of a closed convex cone if it starts there. These viability theorems are the common ancestor between the linear algebraic (optimization) and Lie theoretic approaches. The background to this story is provided in more detail in Section 3 after we introduce the necessary notation.

## 2 Definitions and notation

Throughout this section, $V$ will be a finite-dimensional real Hilbert space. The set of all linear operators on $V$ is $\mathcal{B}(V)$, and every $L \in \mathcal{B}(V)$ has an adjoint $L^* \in \mathcal{B}(V)$ such that $\langle L(x), y\rangle = \langle x, L^*(y)\rangle$ for all $x, y \in V$. If $L \in \mathcal{B}(V)$ is invertible and if $L(X) = X$, then $L$ is an automorphism of $X \subseteq V$ and we write $L \in \operatorname{Aut}(X)$. The space $\mathbb{R}^n$ has the usual inner product and nonnegative orthant $\mathbb{R}^n_+ \coloneqq \{x \in \mathbb{R}^n \mid x_i \geq 0 \text{ for all } i\}$. The real identity matrix of the appropriate size is denoted by $I$. The topological boundary of a set $X$ is $\operatorname{bdy}(X)$, and the topological closure of $X$ is $\operatorname{cl}(X)$.

**Definition 1.** A nonempty subset $K$ of $V$ is a *cone* if $\lambda K \subseteq K$ for all $\lambda \geq 0$. A *closed convex cone* is a cone that is closed and convex as a subset of $V$.

**Definition 2.** A convex cone $K$ in $V$ is *solid* if $\dim(\operatorname{span}(K)) = \dim(V)$. The *lineality space* of a convex cone $K$ is $\operatorname{linspace}(K) := -K \cap K$ and $K$ is *pointed* if $\dim(\operatorname{linspace}(K)) = 0$. A pointed, solid, and closed convex cone is *proper*.

**Definition 3.** If $K$ is a subset of $V$, then the *dual cone* of $K$ is

$$K^* := \{y \in V \mid \langle x, y \rangle \geq 0 \text{ for all } x \in K\}.$$

**Definition 4.** An operator $L \in \mathcal{B}(V)$ is a *positive operator* on $K \subseteq V$ if $L(K) \subseteq K$. The set of all such operators is denoted by $\pi(K)$. We say that $L$ is *exponentially-positive on* $K$ if $e^{tL} \in \pi(K)$ for all $t \geq 0$.

Note that some authors use the (perhaps more accurate) term *exponential-nonnegativity* to refer to the same concept. For better or worse, we will consistently use "exponential-positivity," which arises in the following manner.

**Example 1.** Let $x'(t) = L(x(t))$ be the dynamical system whose solution is $x(t) = e^{tL}(x(0))$. If $L$ is exponentially-positive on a closed convex cone $K$ and if $x(0) \in K$, then $x(t) \in K$ for all $t \geq 0$.

**Definition 5.** The *complementarity set* of $K$ is

$$C(K) := \{(x, s) \in K \times K^* \mid \langle x, s \rangle = 0\}.$$

**Definition 6.** If $K$ is a closed convex set and if $x \in \operatorname{bdy}(K)$, then *the outward normal cone to* $K$ *at* $x$ is

$$N_K(x) := \{v \in V \mid \langle v, y - x \rangle \leq 0 \text{ for all } y \in K\}.$$

If $v \neq 0$ belongs to $N_K(x)$, then $v$ is an outward-pointing (with respect to $K$) normal vector to some hyperplane that supports $K$.

**Definition 7.** If $K$ is a closed convex set in $V$ and if $x \in \operatorname{bdy}(K)$, then we say that $z \in V$ is *subtangential to* $K$ *at* $x$ if $z \in -N_K(x)^*$.

The relevance of the previous two definitions is that—intuitively—if $z$ is subtangential to $K$ at $x$, then the direction $z$ "points into $K$" from $x$.

**Definition 8.** An operator $L \in \mathcal{B}(V)$ is a ***Z**-operator* on $K \subseteq V$ if

$$\langle L(x), s \rangle \leq 0 \text{ for all } (x, s) \in C(K). \tag{1}$$

By $\mathbf{Z}(K)$ we denote the set of all **Z**-operators on $K$. A ***Z**-matrix* is an element of $\mathbf{Z}\left(\mathbb{R}^n_+\right)$.

By considering pairs $(x, s)$ of distinct standard basis vectors in (1), one easily sees that $\mathbf{Z}\left(\mathbb{R}^n_+\right)$ is precisely the set of real $n \times n$ matrices with nonpositive off-diagonal elements. Any **Z**-matrix can thus be written as $\lambda I - N$ where $N \in \pi\left(\mathbb{R}^n_+\right)$, or equivalently where $N$ has nonnegative entries.

**Definition 9.** A real square matrix $A$ is an **M**-matrix if there exist $N \in \pi\left(\mathbb{R}^n_+\right)$ with spectral radius $\rho(N)$ and $\lambda \geq \rho(N)$ such that $A = \lambda I - N$.

Thus, all **M**-matrices are **Z**-matrices. The set $\mathbf{Z}(K)$ is a closed convex cone and it contains the subspace of Lyapunov-like operators.

**Definition 10.** An operator $L \in \mathcal{B}(V)$ is *Lyapunov-like* on $K \subseteq V$ if

$$\langle L(x), s\rangle = 0 \text{ for all } (x, s) \in C(K).$$

By $\mathbf{LL}(K)$ we denote the set of all Lyapunov-like operators on $K$.

The set $\mathbf{LL}(K)$ is a vector space and $\mathbf{LL}(K) = \operatorname{linspace}(\mathbf{Z}(K))$. It is interesting to note that $\mathbf{LL}(K)$ is the Lie algebra of the automorphism group of $K$ [16].

# 3 Timeline

1937 Ostrowski [18] introduces the concept of an "**M**-determinant," named after **M**inkowski. An **M**-determinant corresponds implicitly to an **M**-matrix, and this is considered by Friedland, Hershkowitz, and Schneider [9] to be the first appearance of **M**-matrices. Ostrowski notes that an **M**-determinant corresponds to the determinant of a matrix of the form $\lambda I - N$, where $N$ is nonnegative.

1942 Nagumo [15] publishes a viability theorem under very general conditions.

1956 Ostrowski [19] uses the term ***M**-matrix*, perhaps for the first time, in reference to a matrix having an **M**-determinant.

1960 Varga [28] formalizes the notion of a regular splitting $M = A - B$ outside of the context of **M**-matrices.

1962 Fiedler and Pták [14] use the letter **Z** to denote what we now call **Z**-matrices. Friedland, Hershkowitz, and Schneider [9] state that this is the origin of the name.

1965 Schneider [22] remarks that the regular splitting $\mathbf{LL}(K) - \pi(K)$ generalizes **Z**-matrices on $K = \mathbb{R}^n_+$.

1967 Yorke [29] rediscovers Nagumo's viability theorem.

1969–1970 Bony [3] and Brezis [5] formulate their own viability theorems that are essentially Nagumo and Yorke's.

**Theorem 1** (Nagumo, Yorke, Bony, Brezis)**.** *Let $K$ be a closed subset of a $\mathcal{C}^2$ manifold $M$. If $L$ is a Lipschitz-continuous vector field defined on $M$, then the following are equivalent:*

*(a) Any integral curve of $L$ that starts in $K$ will remain in $K$.*

*(b) $\langle L(x), s\rangle \leq 0$ for any exterior normal vector $s$ to $x \in K$.*

1970 Schneider and Vidyasagar [23] introduce the class of *cross-positive* operators on a proper cone $K$. The authors prove that an operator is cross-positive on $K$ if and only if it is exponentially-positive on $K$. However, the proof relies heavily on the fact that $K$ is proper, and provides little geometric intuition.

The authors also show that $\mathbf{Z}(K)$ is not equal to $\lambda I - \pi(K)$ in general. In particular this shows that the earlier regular splitting of Schneider might not be the best way to define general $\mathbf{Z}$-operators.

1981 Stern [24] investigates the relationship between $K$-regularity, where $L \in \pi(K) - \lambda I$, and exponential-positivity on $K$ in the context of generalized $\mathbf{M}$-matrices. Considering that classical $\mathbf{M}$-matrices are just $\mathbf{Z}$-matrices with an additional property, Stern is (among other things) comparing two candidate generalizations of $\mathbf{Z}$-matrices here.

1982 By specializing Theorem 1 to closed convex sets in a vector space and a linear operator, Stern [25] shows that subtangentiality is both necessary and sufficient for exponential-positivity.

**Theorem 2.** *If $K$ is a closed convex cone in a finite-dimensional real Hilbert space $V$ and if $L \in \mathcal{B}(V)$, then $e^{-tL} \in \pi(K)$ for all $t \geq 0$ if and only if $-L(x)$ is subtangential to $K$ at every $x \in \operatorname{bdy}(K)$.*

1986 Hilgert and Hofmann [13] specialize the results of Bony and Brezis to obtain a general analogue of Stern's 1982 result. The second condition below is essentially the defining property of a $\mathbf{Z}$ operator; later we will see that it is equivalent to Stern's subtangentiality condition in Theorem 2.

**Theorem 3.** *Let $K$ be a closed convex cone in a finite-dimensional real Hilbert space $V$. If $L : V \to V$ is a Lipschitz-continuous vector field with associated flow $F_t$, then the following are equivalent:*

*(a)* *$K$ is invariant under all $F_t$ with $t \geq 0$.*
*(b)* *$\langle L(x), s\rangle \geq 0$ for all $x \in K$ and $s \in K^* \cap \operatorname{span}(\{x\})^{\perp}$.*
*(c)* *$L(x) \in \operatorname{cl}(K + \operatorname{span}(x))$ for all $x \in K$.*

The authors then specialize this result to the linear case to derive equivalent conditions similar to the ones obtained much later by Orlitzky [17].

1987 Stern and Tsatsomeros [26] continue the work that was begun by Stern [24] and investigate the properties of generalized $\mathbf{M}$-matrices. When the $\mathbf{Z}$-matrix property is replaced by $K$-regularity, the result is called a $K$-general $M$-matrix. When the $\mathbf{Z}$-property is replaced by exponential-positivity, the result is called a $K$-extended $\mathbf{M}$-matrix.

2004 Damm [8] uses exponential-positivity to show that the Lyapunov transformations in dynamical systems are precisely those operators $L$ such that both $L$ and $-L$ are cross-positive (a la Schneider and Vidyasagar) on the cone of symmetric/Hermitian positive-semidefinite matrices.

2006 In a paper that was ultimately published in 2009, Gowda and Tao [11] formulate our Definition 8 of a **Z**-operator in the special case where the cone is proper. The authors study **Z**-operators whose eigenvalues are all strictly positive, which, in the terminology of Stern and Tsatsomeros, are invertible $K$-extended **M**-matrices.

2007 Gowda and Sznajder [10] formulate Definition 10 of Lyapunov-like operators in the special case of symmetric (self-dual and homogeneous) cones.

2011 Rudolf, Noyan, Papp, and Alizadeh [21] introduce the bilinearity rank of a proper cone as the dimension of a space of bilinear complementarity relations. The bilinearity rank of $K$ in some sense quantifies how easy it is to solve a complementarity problem over $K$ by splitting the "complementary slackness" equation into a system of multiple equations.

2014 Gowda and Tao [12] notice that the space of bilinear complementarity relations is isomorphic to the space of Lyapunov-like operators (now defined for a proper—and not necessarily symmetric—cone), and coin the term *Lyapunov rank* for its dimension.

2017 Orlitzky [16] shows that the concepts of Lyapunov-like operator and Lyapunov rank are meaningful for closed convex cones in general. This simplifies the practical computation of $\mathbf{LL}(K)$ when $K$ is polyhedral.

2018 Orlitzky [17] shows that the Gowda/Tao definition of a **Z**-operator and the associated exponential-positivity extends to a closed convex cone. In particular, we have the following.

**Theorem 4.** *If $K$ is a closed convex cone in a finite-dimensional real Hilbert space $V$ and if $L \in \mathcal{B}(V)$, then $L \in \mathbf{Z}(K)$ if and only if $e^{-tL} \in \pi(K)$ for all $t \geq 0$.*

# 4 Connecting Z-operators to subtangentiality

From Theorems 2 and 4, we can draw the conclusion that $L \in \mathbf{Z}(K)$ if and only if $-L(x)$ is subtangential to $K$ at every $x \in \operatorname{bdy}(K)$. But why? We would like to prove this fact directly, showing that there is a simple geometric principle underlying the definition of a **Z**-operator. (Recall that the definition of a **Z**-operator goes back to Schneider and Vidyasagar, whose proof was not geometric and worked only for proper cones.) To do this, we need only a few minor results. The following proposition combines Ben-Israel's Theorem 1.3 and Corollary 1.7 [1], and Rockafellar's Theorem 14.6 [20].

**Proposition 1.** *If $K, J$ are closed convex cones, then $(K+J)^{*} = K^{*} \cap J^{*}$ and $\operatorname{cl}(K^{*}+J^{*}) = (K \cap J)^{*}$. Moreover we have that $\operatorname{linspace}(K) = \operatorname{span}(K^{*})^{\perp}$.*

**Lemma 1** (Stern [25], Lemma 2.7)**.** *If $K$ is a closed convex cone in a finite-dimensional real Hilbert space $V$ and if $x \in \operatorname{bdy}(K)$, then $\langle v, x\rangle = 0$ for all $v \in N_{K}(x)$.*

*Proof.* Substitute $y = 0$ and $y = 2x$ into Definition 6 to find $\langle v, \pm x \rangle \leq 0$. ☐

The next result appears as Lemma 1.4 in Hilgert and Hofmann [13], but follows easily from Lemma 1.

**Corollary 1.** *If $K$ is a closed convex cone in a finite-dimensional real Hilbert space $V$ and if $x \in \operatorname{bdy}(K)$, then $N_K(x) = -K^* \cap \operatorname{span}(\{x\})^\perp$.*

Observe that $N_K(x) \subseteq \operatorname{bdy}(-K^*)$ in Corollary 1: if $v \in -K^*$ and if $\langle v, \pm x \rangle = 0$ for $x \in K$, then it is apparent that $v$ is orthogonal to some element of $-K$, and thus that $v \in \operatorname{bdy}(-K^*) = -\operatorname{bdy}(K^*)$.

We digress briefly to mention some related results. A subcone $F$ of $K$ is called a *face* of $K$ if $x, y \in K$ and $x + y \in F$ together imply that $x, y \in F$. The duality operator [27] of a closed convex cone $K$ is defined on the faces $F$ of $K$ by $d_K(F) = K^* \cap \operatorname{span}(F)^\perp$. In particular when $F = \operatorname{cone}(\{x\})$, we have

$$d_K(\operatorname{cone}(\{x\})) = K^* \cap \operatorname{span}(\operatorname{cone}(\{x\}))^\perp = -N_K(x).$$

This is not a new observation; Tam himself notes the equality. Corollary 1 is of interest because for $z$ to be subtangential to $K$ at $x$ means that $-z \in N_K(x)^*$. We also record the following from Lemma 1.4 in Hilgert and Hofmann [13], which follows immediately from the definition of subtangentiality and Proposition 1.

**Proposition 2.** *If $K$ is a closed convex cone in a finite-dimensional real Hilbert space $V$ and if $x \in \operatorname{bdy}(K)$, then $z$ is subtangential to $K$ at $x$ if and only if $z \in \operatorname{cl}(K + \operatorname{span}(\{x\}))$.*

Tam's Proposition 3.1 [27] shows that cones of the form $K + \operatorname{span}(\{x\})$ are called a *point's cone*, or *the cone of $K$ at $x$*, and they are of particular interest. After taking the closure, $\operatorname{cl}(K + \operatorname{span}(\{x\}))$ is called the *cone of support of $K$ at $x$*. It is known that every proper non-polyhedral cone $K$ has some $x$ such that $K + \operatorname{span}(\{x\})$ is not closed, and that therefore the closure in Proposition 2 is not superfluous. The next result also follows immediately from Corollary 1.

**Lemma 2.** *If $K$ is a closed convex cone in a finite-dimensional real Hilbert space $V$ and if $x \in \operatorname{bdy}(K)$, then*

$$(x, s) \in C(K) \iff -s \in N_K(x).$$

**Lemma 3.** *If $K$ is a closed convex cone in a finite-dimensional real Hilbert space $V$ and if $L \in \mathcal{B}(V)$, then $L \in \mathbf{Z}(K)$ if and only if $-L(x)$ is subtangential to $K$ at every $x \in \operatorname{bdy}(K)$.*

*Proof.* The following are equivalent after an application of Lemma 2:

(a) $L \in \mathbf{Z}(K)$.

(b) $\langle L(x), s \rangle \leq 0$ for all $(x, s) \in C(K)$.

(c) $\langle L(x), s \rangle \leq 0$ for all $x \in \operatorname{bdy}(K)$ and $s \in -N_K(x)$.

(d) $-L(x) \in -N_K(x)^*$ for all $x \in \operatorname{bdy}(K)$.

(e) $-L(x)$ is subtangential to $K$ at $x$ for all $x \in \operatorname{bdy}(K)$. □

Recall that $-L(x)$ being subtangential to $K$ at $x$ has the intuitive interpretation that $-L(x)$ "points into $K$" from the point $x$. Consider this in the context of Example 1: if $x'(t) = -L(x)$ points into the cone $K$ at every boundary point $x$ of $K$, then obviously $x(t)$ will remain in $K$ for all $t \geq 0$ given that $x(0)$ starts there. One need only note that $x(t) = e^{-tL}(x(0))$ to deduce exponential-positivity in that case.

Thus, Lemma 3 gives us an intuitive explanation for why Definition 8 results in exponential-positivity on a closed convex cone. And it does so directly, without having to appeal to the result for proper cones. We now merely combine known results into a new form.

**Theorem 5.** *If $K$ is a closed convex cone in a finite-dimensional real Hilbert space $V$ and if $L \in \mathcal{B}(V)$, then the following are equivalent:*

*(a)* $L \in \mathbf{Z}(K)$.

*(b)* $-L(x)$ *is subtangential to* $K$ *at every* $x \in \operatorname{bdy}(K)$.

*(c)* $-L(x) \in \operatorname{cl}(K + \operatorname{span}(\{x\}))$ *for all* $x \in \operatorname{bdy}(K)$.

*Proof.* The first two items are equivalent by Lemma 3, and the last is equivalent by Proposition 2. □

Theorem 5 incorporates one item from each of Theorems 2, 3 and 4. More in-depth equivalent conditions have been published by each group of authors. Having different perspectives on a single idea is of inherent value. Nevertheless, to further justify this endeavor we demonstrate a simpler proof of the following Theorem 2 in Gowda and Tao [12].

**Theorem.** *If $K$ is a proper polyhedral cone in a finite-dimensional real Hilbert space $V$ and if $L \in \mathcal{B}(V)$, then $L \in \mathbf{LL}(K)$ if and only if each $x \in \operatorname{Ext}(K)$ is an eigenvector of $L$.*

*Proof.* It is already known [16] that if $L$ is Lyapunov-like on the generators (in this case, $\operatorname{Ext}(K)$) of $K$, then $L$ is Lyapunov-like on all of $K$. So we will prove only the difficult implication, namely that if $L \in \mathbf{LL}(K)$, then each $x \in \operatorname{Ext}(K)$ is an eigenvector of $L$.

From the definition $\mathbf{LL}(K) := -\mathbf{Z}(K) \cap \mathbf{Z}(K)$ and the last item in Theorem 5, we see that $L \in \mathbf{LL}(K)$ implies $L(x) \in \operatorname{linspace}(\operatorname{cl}(K + \operatorname{span}(\{x\})))$ for all $x \in \operatorname{bdy}(K)$, and in particular for all $x \in \operatorname{Ext}(K)$. Since $K$ is polyhedral, we have $\operatorname{cl}(K + \operatorname{span}(\{x\})) = K + \operatorname{span}(\{x\})$. And Tam [27] has proven that $\operatorname{linspace}(K + \operatorname{span}(\{x\})) = \operatorname{span}(\phi(\{x\}))$, where $\phi(\{x\})$ is the minimal face of $K$ containing $\{x\}$. However for any $x \in \operatorname{Ext}(K)$, we have that $\operatorname{cone}(\{x\})$ is a (necessarily minimal) face of $K$, and thus that $\operatorname{linspace}(K + \operatorname{span}(\{x\})) = \operatorname{span}(\{x\})$. Combining everything, we arrive at

$$L \in \mathbf{LL}(K) \implies L(x) \in \operatorname{span}(\{x\}) \text{ for all } x \in \operatorname{Ext}(K). \qquad \square$$

## 5 Some questions

*Question* 1. Can Theorem 5 be used to prove that $L_1 \circ L_2 - L_2 \circ L_1$ is Lyapunov-like on a given cone $K$ whenever $L_1$ and $L_2$ are? This is already known to be true, since $\mathbf{LL}(K)$ is a Lie algebra. However, based solely on Definition 10, this result looks like magic.

*Question* 2. The cone of support to $K$ at $x$ is not generally equal to $K + \operatorname{span}(\{x\})$. Can this be used to produce a new proof of $\mathbf{Z}(K) \neq \mathbf{LL}(K) - \pi(K)$?

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