Positive and **Z**-operators on closed convex cones

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Abstract

Let K be a closed convex cone with dual K^* in a finite-dimensional real Hilbert space. A *positive operator* on K is a linear operator L such that $L(K) \subseteq K$. Positive operators generalize the nonnegative matrices and are essential to the Perron-Frobenius theory. We say that L is a ***Z**-operator* on K if

$$\langle L(x), s \rangle \leq 0 \text{ for all } (x, s) \in K \times K^* \text{ such that } \langle x, s \rangle = 0.$$

The **Z**-operators are generalizations of **Z**-matrices (whose off-diagonal elements are nonpositive) and they arise in dynamical systems, economics, game theory, and elsewhere. We connect the positive and **Z**-operators. This extends the work of Schneider, Vidyasagar, and Tam on proper cones, and reveals some interesting similarities between the two families.

1 Introduction

Positive operators arose from the study of integral operators and matrices with nonnegative entries [1]. Perron showed that a matrix with positive entries has a simple eigenvalue equal to its spectral radius and that some corresponding eigenvector has positive entries. Moreover its other eigenvalues are strictly less than the spectral radius in modulus. Frobenius partially extended Perron's result to nonnegative matrices, and the nonnegative matrices are positive operators in that setting.

Suppose that V is an ordered vector space and that $x \geq 0$ in V. In the theory of operators [1], x is called a *positive element* of V. A *positive operator* is a linear operator that sends positive elements of V to positive elements. Every proper cone K orders [3] its ambient space by $x \geq 0 \iff x \in K$. With respect to this ordering, we denote the set of positive operators by

$$\pi(K) \coloneqq \{L : V \to V \mid L \text{ is linear and } L(K) \subseteq K\}.$$

The Perron-Frobenius theorem is thus a statement about positive operators on the cone $K = \mathbb{R}^n_+$, the nonnegative orthant in $\mathbb{R}^n$. The Krein-Rutman theorem extends Perron-Frobenius to a compact positive linear operator with positive spectral radius on a Banach space ordered by a closed convex pointed cone.

A $\mathbf{Z}$-matrix is a real square matrix whose off-diagonal entries are nonpositive. Equivalently, a $\mathbf{Z}$-matrix has the form $\lambda I - N$ where N is a nonnegative matrix (that is, a positive operator on $\mathbb{R}^n_+$). It is therefore not surprising that the two theories are intertwined. Berman and Plemmons [4] cite an astounding number of equivalent conditions for $\mathbf{Z}$-matrices to be nonsingular $\mathbf{M}$-matrices, connecting them to many different areas.

Generalizations of $\mathbf{Z}$-matrices have started to appear [5, 6]. Our definition of a $\mathbf{Z}$-operator is due to Gowda and Tao [11]. If K^* represents the dual of K, then L is a $\mathbf{Z}$-operator on K and we write $L \in \mathbf{Z}(K)$ if $\langle L(x), s\rangle \leq 0$ for all $x \in K$ and $s \in K^*$ such that $\langle x, s\rangle = 0$. This definition reduces to that of a $\mathbf{Z}$-matrix when $K = \mathbb{R}^n_+$. These $\mathbf{Z}$-operators emerge in dynamical systems [8, 11], complementarity problems [11], game theory [9], economics, and everywhere that $\mathbf{Z}$-matrices arise [4]. Kuzma et al. [15] recently resolved an open problem that applies $\mathbf{Z}$-operators to mathematical finance. In each of these cases, the cone K is assumed to be *proper*: closed, convex, pointed, and solid.

Schneider and Vidyasagar [21] and Elsner [8] discovered a striking connection between the positive and $\mathbf{Z}$-operators on a proper cone. We eventually extend this result to any closed convex cone in finite dimensions.

Theorem. *If K is a proper cone in $\mathbb{R}^n$ and if A is a matrix in $\mathbb{R}^{n\times n}$, then $A \in \mathbf{Z}(K)$ if and only if $e^{-tA} \in \pi(K)$ for all $t \geq 0$.*

The set of all $\mathbf{Z}$-operators contains a subspace $\mathbf{LL}(K) \coloneqq -\mathbf{Z}(K) \cap \mathbf{Z}(K)$ of *Lyapunov-like* operators. Lyapunov-like operators are important because they can be used to solve the equation $\langle x, s\rangle = 0$ for $x \in K$ and $s \in K^*$ that appears as optimality conditions in convex optimization [19]. One motivation for studying $\mathbf{Z}$-operators is their connection to the Lyapunov-like operators. Implicit in the work of Schneider and Vidyasagar is the following.

Theorem. *If K is a proper cone in $\mathbb{R}^n$, then $\mathbf{Z}(K) = \operatorname{cl}(\mathbf{LL}(K) - \pi(K))$.*

We will also generalize this result. Sometimes the closure is superfluous and $\mathbf{Z}(K) = \mathbf{LL}(K) - \pi(K)$; the problem solved by Kuzma et al. was of that type. By studying $\mathbf{Z}(K)$ and $\pi(K)$, we hope to gain insight into similar problems.

There is also a practical motivation for extending these results to closed convex cones. To compute $\pi(K)$ or $\mathbf{Z}(K)$, we need a representation of the cone K that can be fed as input into an algorithm. There is a natural way to represent a polyhedral convex cone since any finite set of vectors can be identified with the cone it generates. As a result, existing algorithms tend to operate on polyhedral convex cones (which are necessarily closed). No similar representation is known for proper polyhedral cones: given a set of vectors, how can one determine if the cone it generates is proper? The best answer to that question currently involves a verification step that we would like to avoid by showing that $\pi(K)$ and $\mathbf{Z}(K)$ are meaningful for all closed convex K. The concept of Lyapunov rank was extended to closed convex cones for similar reasons [16].

Theorems 1 and 4 provide generators of $\pi(K)^*$ and $\mathbf{Z}(K)^*$. When K is polyhedral, Algorithms 1 and 2 turn those theorems into a method for computing

$\pi(K)$ and $\mathbf{Z}(K)$ using exact rational arithmetic. In Section 2.4, we introduce an isometry that associates a proper cone to every closed convex cone. However, that isometry will often involve irrational roots and thus inexact arithmetic. In other words, one cannot simply apply the isometry and fall back on the known algorithms for proper cones. It is because they avoid that issue that our new algorithms—implemented in the SageMath [28] system—are useful.

2 Preliminaries

Throughout this section, V will be a finite-dimensional real Hilbert space. Here and from now on, the term "Euclidean space" will be used for such a space.

2.1 Standard definitions

Let W be another Euclidean space. The set of all linear operators from V to W forms a vector space which we denote by $\mathcal{B}(V, W)$. We abbreviate $\mathcal{B}(V, V)$ by $\mathcal{B}(V)$. If $L \in \mathcal{B}(V, W)$ is invertible and preserves inner products, then L is an *isometry*. Any $L \in \mathcal{B}(V, W)$ has an adjoint $L^* \in \mathcal{B}(W, V)$ such that $\langle L(x), y\rangle = \langle x, L^*(y)\rangle$ for all $x \in V$ and $y \in W$. The identity operator on V is $\mathrm{id}_V \in \mathcal{B}(V)$. Given two elements x and s in V, we define $s \otimes x$ to be the operator $y \mapsto \langle x, y\rangle s$ on V. For subsets S and X of V, we will write $S \otimes X :=$ $\{s \otimes x \mid s \in S, x \in X\}$. The adjoint of $s \otimes x$ is $x \otimes s$, and $s \otimes L^*(x) = (s \otimes x) L$ is the composition of the operators $s \otimes x$ and $L \in \mathcal{B}(V)$.

Define the trace operator on $\mathcal{B}(V)$ to be the sum-of-eigenvalues, $\operatorname{trace}(L) := \sum_{\lambda \in \sigma(L)} \lambda$. Then $\langle L_1, L_2\rangle := \operatorname{trace}(L_1 \circ L_2^*)$ is our inner product on linear operator spaces. Later we use the fact that $\operatorname{trace}(s \otimes x) = \operatorname{trace}(x \otimes s) = \langle x, s\rangle$. The topological closure of a set X is $\operatorname{cl}(X)$.

If W is a subspace of V, then the *orthogonal complement* of W is another subspace of V defined by $W^{\perp} := \{y \in V \mid \langle x, y\rangle = 0 \text{ for all } x \in W\}$, and V has direct sum decomposition $V = W \oplus W^{\perp}$. If $\mathcal{E}, \mathcal{F}, \mathcal{G}$, and $\mathcal{H}$ are sets of linear operators whose domains and codomains are such that it makes sense to do so, then we will use the shorthand notation

$$\begin{bmatrix} \mathcal{E} & \mathcal{F} \\ \mathcal{G} & \mathcal{H} \end{bmatrix} := \left\{ \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \;\middle|\; \begin{matrix} L_{11} \in \mathcal{E}, L_{12} \in \mathcal{F} \\ L_{21} \in \mathcal{G}, L_{22} \in \mathcal{H} \end{matrix} \right\}$$

to denote a set of block-form operators.

The real n-space $\mathbb{R}^n$ is equipped with the usual inner product, standard basis $(e_1, e_2, \ldots, e_n)$, and nonnegative orthant $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x_i \geq 0 \text{ for all } i\}$. The real identity matrix of the appropriate size is denoted by I.

2.2 Cone definitions

Definition 1. A nonempty subset K of V is a *cone* if $\lambda K \subseteq K$ for all $\lambda \geq 0$. A *closed convex cone* is a cone that is closed and convex as a subset of V.

Definition 2. The *conic hull* of a nonempty subset X of V is a convex cone,

$$\operatorname{cone}(X) := \left\{ \sum_{i=1}^{m} \alpha_i x_i \;\middle|\; x_i \in X,\ \alpha_i \geq 0,\ m \in \mathbb{N} \right\}.$$

Definition 3. If $\operatorname{cone}(G) = K$, then G *generates* K and the elements of G are *generators* of K. If a finite set generates K, then K is *polyhedral*.

Clasically, a polyhedral cone is defined to be the finite intersection of homogeneous half-spaces. However, Theorem 1.3 in Ziegler [29] or Theorem 19.1 in Rockafellar [17] shows that the two definitions are equivalent. For cones, the two implications in that equivalence are known as the theorems of Minkowski and Weyl, and can also be found as Theorem 2.8.6 and Theorem 2.8.8 in Stoer and Witzgall [24]. As a result of the equivalence, all polyhedral cones are closed [2].

Definition 4. The dimension of $K \subseteq V$ is $\dim(K) := \dim(\operatorname{span}(K))$. A convex cone K in V is *solid* if it has nonempty interior in V, or, equivalently, if $\dim(K) = \dim(V)$. The *lineality space* of a convex cone K is $\operatorname{linspace}(K) := -K \cap K$. Its *lineality* is $\operatorname{lin}(K) := \dim(\operatorname{linspace}(K))$, and K is *pointed* if $\operatorname{lin}(K) = 0$. A pointed, solid, and closed convex cone is *proper*.

If S and X are linearly-independent subsets of V, then $S \otimes X$ is linearly-independent in $\mathcal{B}(V)$, and it follows that $\dim(S \otimes X) = \dim(S)\dim(X)$ [18]. Proper cones have a convenient set of generators that are, in a sense, minimal.

Definition 5. An element x in the convex cone K is an *extreme vector* of K if it is not a positive linear combination of two linearly-independent vectors in K. The set of all unit-norm extreme vectors of K is $\operatorname{Ext}(K)$.

Extreme vectors are primarily used with proper cones. If K is closed and pointed, then $K = \operatorname{cone}(\operatorname{Ext}(K))$ by a version of the Krein-Milman theorem.

Definition 6. If K is a subset of V, then the *dual cone* K^* of K is given by

$$K^* := \{ y \in V \mid \langle x, y \rangle \geq 0 \text{ for all } x \in K \}.$$

The following dual relationships are well-known and will be used freely. They may be found in Ben-Israel's [2] Theorem 1.3, Theorem 1.5, and Corollary 1.6.

- The dual K^* is a closed convex cone for any subset $K \subseteq V$.
- If K is a convex cone, then $(K^*)^* = \operatorname{cl}(K)$.
- A subset $K \subseteq V$ is a closed convex cone in V if and only if $(K^*)^* = K$.

Rockafellar's Corollary 19.2.2 relates duality to polyhedrality [17].

Proposition 1. *If K is a polyhedral convex cone, then K^* is polyhedral. A closed convex cone K is polyhedral if and only if its dual K^* is polyhedral.*

Duals are defined in terms of the inner product, so they are preserved under isometry: $\psi(K)^* = \psi(K^*)$ for every isometry ψ. Likewise, the dual of a cartesian product $(K_1 \times K_2)^*$ is the cartesian product of duals $K_1^* \times K_2^*$. The following proposition combines Ben-Israel's Theorem 1.3 and Corollary 1.7 [2].

Proposition 2. *If K and J are closed convex cones, then $(K + J)^* = K^* \cap J^*$ and $\operatorname{cl}(K^* + J^*) = (K \cap J)^*$.*

Finally, we will need Rockafellar's Theorem 14.6 for the duality between pointed and solid cones [17].

Proposition 3. *If K is a closed convex cone, then $\operatorname{linspace}(K) = \operatorname{span}(K^*)^{\perp}$.*

2.3 Classes of linear operators

Our main results concern a few classes of linear operators. They are all sensibly defined on any subset $K \subseteq V$, but in practice, K will be a closed convex cone.

Definition 7. An operator $L \in \mathcal{B}(V)$ is a *positive operator* on $K \subseteq V$ if $L(K) \subseteq K$. The set of all such operators is denoted by $\pi(K)$.

The prototypical positive operators are nonnegative matrices [4] on $K = \mathbb{R}^n_+$. If K is a closed convex cone, then we have an alternative characterization:

$$L \in \pi(K) \iff \langle L(x), s \rangle \geq 0 \text{ for all } (x, s) \in K \times K^*.$$

The requisite property of a $\mathbf{Z}$-operator is similar, but it need only hold on pairs of orthogonal vectors in $K \times K^*$.

Definition 8. The *complementarity set* of K is

$$C(K) := \{(x, s) \in K \times K^* \mid \langle x, s \rangle = 0\}.$$

Definition 9. An operator $L \in \mathcal{B}(V)$ is a ***Z**-operator* on $K \subseteq V$ if

$$\langle L(x), s \rangle \leq 0 \text{ for all } (x, s) \in C(K).$$

By $\mathbf{Z}(K)$ we denote the set of all $\mathbf{Z}$-operators on K.

When $K = \mathbb{R}^n_+$, the complementarity set $C\left(\mathbb{R}^n_+\right)$ contains all pairs of distinct standard basis vectors. The requirement on $\mathbf{Z}\left(\mathbb{R}^n_+\right)$ gives rise to matrices whose off-diagonal elements are nonpositive—the $\mathbf{Z}$-matrices. The set $\mathbf{Z}(K)$ is a closed convex cone and it contains the subspace of Lyapunov-like operators.

Definition 10. An operator $L \in \mathcal{B}(V)$ is *Lyapunov-like* on $K \subseteq V$ if

$$\langle L(x), s \rangle = 0 \text{ for all } (x, s) \in C(K).$$

By $\mathbf{LL}(K)$ we denote the set of all Lyapunov-like operators on K.

The set $\mathbf{LL}(K)$ is a vector space and $\mathbf{LL}(K) = \operatorname{linspace}(\mathbf{Z}(K))$. Finding Lyapunov-like operators is an interesting problem. The search began with Rudolf et al. [19] and has been continued by others [10, 12, 16].

2.4 Decomposing improper cones

Any closed convex cone is isometric to a cartesian product of a proper cone, a subspace, and a trivial cone. The following is well-known and appears, for example, as Stoer and Witzgall's Theorem 2.10.5 [24].

Proposition 4. *If K is a convex cone in a Euclidean space, then K has an orthogonal direct sum decomposition into two convex cones,*

$$K = K \cap \operatorname{linspace}(K)^{\perp} \oplus \operatorname{linspace}(K).$$

Its first factor $K \cap \operatorname{linspace}(K)^{\perp}$ is pointed.

Observe that any convex cone K is solid in the ambient space $\operatorname{span}(K)$, and that $K \cap \operatorname{linspace}(K)^{\perp}$ is pointed by Proposition 4. If K_p represents the cone $K \cap \operatorname{linspace}(K)^{\perp}$ living in the subspace $\operatorname{span}(K) \cap \operatorname{linspace}(K)^{\perp}$, then K_p is both solid and pointed. Now, the ambient space V is an orthogonal direct sum,

$$V = \operatorname{linspace}(K) \oplus \operatorname{span}(K) \cap \operatorname{linspace}(K)^{\perp} \oplus \operatorname{span}(K)^{\perp}.$$

From this and Proposition 4 we deduce the existence of a useful isometry of V.

Lemma 1. *If K is a closed convex cone in a Euclidean space V and if $K_p = K \cap \operatorname{linspace}(K)^{\perp}$, then there is an isometry ϕ such that*

$$\begin{aligned} &\phi : V \to V_1 \times V_2 \times V_3 \\ &\phi(K) = \operatorname{linspace}(K) \times K_p \times \{0\} \end{aligned}$$

where

$$\begin{aligned} V_1 &\coloneqq \operatorname{linspace}(K) \\ V_2 &\coloneqq \operatorname{span}(K) \cap \operatorname{linspace}(K)^{\perp} \\ V_3 &\coloneqq \operatorname{span}(K)^{\perp}, \end{aligned}$$

and K_p is a proper cone in V_2. An operator $L \in \mathcal{B}(\phi(V))$ will thus have the block form $L = [L_{ij}]$ where $L_{ij} \in \mathcal{B}(V_j, V_i)$. We abbreviate this as

$$L \in \begin{bmatrix} \mathcal{B}(V_1, V_1) & \mathcal{B}(V_2, V_1) & \mathcal{B}(V_3, V_1) \\ \mathcal{B}(V_1, V_2) & \mathcal{B}(V_2, V_2) & \mathcal{B}(V_3, V_2) \\ \mathcal{B}(V_1, V_3) & \mathcal{B}(V_2, V_3) & \mathcal{B}(V_3, V_3) \end{bmatrix}.$$

This isometry will often reduce our problems to the case of a proper cone, where existing results can be applied.

Proposition 5. *If K is a closed convex cone in V and if ϕ and K_p are as in Lemma 1, then K is polyhedral if and only if K_p is polyhedral.*

Proof. By properties of isometry, we have that K is polyhedral if and only if $\phi(K)$ is polyhedral. The other two factors $\operatorname{linspace}(K)$ and $\{0\}$ in $\phi(K)$ are polyhedral; therefore polyhedrality of $\phi(K)$ depends entirely on that of K_p. $\square$

To reason about the positive and $\mathbf{Z}$-operators in the product space we will use the following two easy results.

Proposition 6. *If K is a closed convex cone in V and if ψ is an isometry of V, then $\mathbf{Z}(\psi(K)) = \psi\mathbf{Z}(K)\psi^{-1}$ and $\pi(\psi(K)) = \psi\pi(K)\psi^{-1}$. As a result, $\pi(\psi(K))$ and $\mathbf{Z}(\psi(K))$ are isometric to $\pi(K)$ and $\mathbf{Z}(K)$ respectively.*

Proposition 7. *If X and S are subsets of V and if ψ is an isometry of V, then $\psi(S \otimes X)\psi^{-1} = \psi(S) \otimes \psi(X)$ is isometric to $S \otimes X$.*

3 Positive operators

Observe that the positive operators on a closed convex cone K themselves form a closed convex cone. The three criteria—that $\pi(K)$ is closed, convex, and a cone—are easy to verify and depend on the same properties of K.

Proposition 8. *If K is a closed convex cone, then so is $\pi(K)$.*

If K is proper, then both $\pi(K)$ and its dual are proper [21]. To determine if some linear operator belongs to $\pi(K)$, it suffices to check positivity on a generating set of K. This can be seen by expanding any element of K in terms of its generators and using the linearity of the operator. However, checking the generators will almost always be impractical if K is not polyhedral.

Proposition 9. *If $K = \operatorname{cone}(G)$ in a Euclidean space V and if $L \in \mathcal{B}(V)$, then $L \in \pi(K)$ if and only if $L(G) \subseteq K$.*

Tam [27] found a simple expression for the generators of the dual of $\pi(K)$ when K is proper. He uses the fact that $\operatorname{cone}(K^* \otimes K)$ is closed to prove that

$$\pi(K)^* = \operatorname{cone}(K^* \otimes K) \text{ if } K \text{ is proper.} \tag{1}$$

These generators also work when K is merely closed and convex. Tam's argument is based on the following equivalence, the conditions of which follow directly from the definitions and a property of the trace.

Proposition 10. *If K is a closed convex cone in a Euclidean space V and if $L \in \mathcal{B}(V)$, then the following are equivalent:*

- $L \in \pi(K)$
- $L(x) \in K$ *for all* $x \in K$
- $\langle L(x), s\rangle \geq 0$ *for all* $x \in K$ *and* $s \in K^*$
- $\langle L, s \otimes x\rangle \geq 0$ *for all* $x \in K$ *and* $s \in K^*$

It follows that $\pi(K)$ is the dual of $\operatorname{cone}(K^* \otimes K)$, and thus that $\pi(K)^* = \operatorname{cl}(\operatorname{cone}(K^* \otimes K))$. Tam shows that $\operatorname{cone}(K^* \otimes K)$ is closed for proper K, and the formula (1) follows. We will take the same approach. Note that any $L \in \operatorname{cone}(K^* \otimes K)$ can be written $L = \sum s_i \otimes x_i$ for $(x_i, s_i) \in K \times K^*$ without scalar factors, since they can be absorbed into $s_i \otimes x_i$.

Lemma 2. *If K is a closed convex cone in a Euclidean space V, then the set* $\operatorname{cone}(K^* \otimes K)$ *is closed.*

Proof. Closedness is preserved under isometry, so let ϕ and K_p be as in Lemma 1. Then $\phi(K)^* = \phi(K^*) = \{0\} \times K_p^* \times \operatorname{span}(K)^\perp$, and Proposition 7 says that instead of $K^* \otimes K$, we can without loss of generality consider

$$\phi(K^*) \otimes \phi(K) = \begin{bmatrix} \{0\} & \{0\} & \{0\} \\ K_p^* \otimes V_1 & K_p^* \otimes K_p & \{0\} \\ V_3 \otimes V_1 & V_3 \otimes K_p & \{0\} \end{bmatrix}.$$

It it straightforward to verify that when all of the sets involved contain zero, the $\operatorname{cone}(\cdot)$ operation acts componentwise. Thus,

$$\operatorname{cone}(\phi(K^*) \otimes \phi(K)) = \begin{bmatrix} \{0\} & \{0\} & \{0\} \\ \operatorname{cone}\left(K_p^* \otimes V_1\right) & \operatorname{cone}\left(K_p^* \otimes K_p\right) & \{0\} \\ \operatorname{cone}(V_3 \otimes V_1) & \operatorname{cone}(V_3 \otimes K_p) & \{0\} \end{bmatrix}.$$

Notice, for example, that

$$\operatorname{cone}\left(K_p^* \otimes V_1\right) = \operatorname{cone}\left(K_p^* \otimes \pm V_1\right) = \operatorname{span}\left(K_p^* \otimes V_1\right),$$

which (by a dimension argument) equals its ambient space $\mathcal{B}(V_1, V_2)$. Using that same reasoning, we obtain

$$\operatorname{cone}(\phi(K^*) \otimes \phi(K)) = \begin{bmatrix} \{0\} & \{0\} & \{0\} \\ \mathcal{B}(V_1, V_2) & \operatorname{cone}\left(K_p^* \otimes K_p\right) & \{0\} \\ \mathcal{B}(V_1, V_3) & \mathcal{B}(V_2, V_3) & \{0\} \end{bmatrix}.$$

Since K_p is proper, we can cite Tam's result for proper cones to conclude that $\operatorname{cone}\left(K_p^* \otimes K_p\right) = \pi(K_p)^*$ is closed. The other sets are obviously closed. ☐

Theorem 1. *If K is a closed convex cone in a Euclidean space, then $\pi(K)^* = \operatorname{cone}(K^* \otimes K)$.*

Proof. Deduce from Proposition 10 that $\pi(K)^* = \operatorname{cl}(\operatorname{cone}(K^* \otimes K))$, and then apply Lemma 2. ☐

This result was known for proper cones, so we look elsewhere for examples.

Example 1. If $K = \operatorname{cone}(\{e_1, \pm e_2\})$ is the right half-space in $V = \mathbb{R}^2$, then $K^* = \operatorname{cone}(\{e_1\})$ and Theorem 1 gives

$$\pi(K)^* = \operatorname{cone}\left(\left\{e_1 e_1^T, \pm e_1 e_2^T\right\}\right).$$

In this simple polyhedral case, we can use the definition of dual cone and the fact that $\pi(K) = \left(\pi(K)^*\right)^*$ to directly compute

$$\pi(K) = \operatorname{cone}\left(\left\{e_1 e_1^T, \pm e_2 e_1^T, \pm e_2 e_2^T\right\}\right).$$

This result is verified using Proposition 9.

Corollary 1. *If K is a closed convex cone in a Euclidean space V and if ϕ and K_p are as in Lemma 1, then*

$$\pi(\phi(K)) = \begin{bmatrix} \mathcal{B}(V_1, V_1) & \mathcal{B}(V_2, V_1) & \mathcal{B}(V_3, V_1) \\ \{0\} & \pi(K_p) & \mathcal{B}(V_3, V_2) \\ \{0\} & \{0\} & \mathcal{B}(V_3, V_3) \end{bmatrix}.$$

Proof. Through Theorem 1, the proof of Lemma 2 gives us a block representation of $\operatorname{cone}(\phi(K^*) \otimes \phi(K)) = \pi(\phi(K))^*$. Simply take duals therein. ☐

We can now extend another result of Tam.

Theorem 2. *If K is a closed convex cone in a Euclidean space V, then $\pi(K)$ is polyhedral if and only if K is polyhedral.*

Proof. Let K_p be as in Lemma 1. Proposition 5 shows that K is polyhedral if and only if K_p is polyhedral. A result of Tam [27] shows that K_p is polyhedral if and only if $\pi(K_p)$ is polyhedral. Combining the two, we deduce that K is polyhedral if and only if $\pi(K_p)$ is polyhedral.

Corollary 1 and Proposition 6 provide the remaining equivalence, that $\pi(K_p)$ is polyhedral if and only if $\pi(K)$ is. This is easily deduced in a manner similar to Proposition 5, since all components of $\pi(\phi(K))$ other than $\pi(K_p)$ are (polyhedral) vector spaces or $\{0\}$. ☐

Lemma 3. *Let K be a closed convex cone in a Euclidean space V, and let $n = \dim(V)$, $m = \dim(K)$, and $\ell = \operatorname{lin}(K)$. Then,*

$$\dim(\pi(K)) = n^2 - \ell(m - \ell) - m(n - m).$$

Proof. Note that $\dim\left(\operatorname{span}(K) \cap \operatorname{linspace}(K)^{\perp}\right) = \dim(K) - \operatorname{lin}(K) = m - \ell$. The result then follows from Corollary 1 and the fact that $\pi(K_p)$ is proper. ☐

Example 2. If $K = \{0\}$ in V, then $m = \ell = 0$, and $\dim(\pi(K)) = n^2$ which agrees with the obvious fact that $\pi(K) = \mathcal{B}(V)$.

Example 3. If K is proper, then in Lemma 3, we have $m = n$ and $\ell = 0$. Thus $\dim(\pi(K)) = n^2$ and $\pi(K)$ is solid.

Example 4. Example 1 has $n = m = 2$ and $\ell = 1$ giving $\dim(\pi(K)) = 3$.

Lemma 4. *If K is a closed convex cone in a Euclidean space V, then*

$$\operatorname{lin}(\pi(K)) = \dim(V)^2 - \dim(K)\dim(K^*).$$

Proof. From Theorem 1, it follows that $\pi(K)^* = \operatorname{cone}(K^* \otimes K)$, whose dimension is $\dim(K)\dim(K^*)$. Therefore, by Proposition 3, we have

$$\operatorname{lin}(\pi(K)) = \dim(V)^2 - \dim\left(\pi(K)^*\right) = \dim(V)^2 - \dim(K)\dim(K^*). \quad ☐$$

Example 5. If $K = \{0\}$ in V, then $\dim(K) = 0$, and $\operatorname{lin}(\pi(K)) = \dim(V)^2$ in agreement with the fact that $\pi(K) = \mathcal{B}(V)$.

Example 6. If K is proper, then $\dim(K) = \dim(K^*) = \dim(V)$. Lemma 4 gives $\operatorname{lin}(\pi(K)) = 0$, showing that $\pi(K)$ is pointed.

Example 7. In Example 1, we have $\operatorname{lin}(\pi(K)) = 4 - 2 \cdot 1 = 2$.

These examples reaffirm that $\pi(K)$ is proper whenever K is proper [20]. In the general setting where K may not be proper, Lemma 4 gives us a converse.

Theorem 3. *If K is a closed convex cone in a Euclidean space, then $\pi(K)$ is proper if and only if K is proper.*

When K is polyhedral, Theorem 1 allows us to compute a generating set of $\pi(K)$. Algorithms to compute the dual generators of a polyhedral cone are known, and the inverse operations $\operatorname{vec}()$ and $\operatorname{mat}()$ are isometries.

Algorithm 1 Compute generators of $\pi(K)$

Input: A polyhedral convex cone K
Output: A generating set of $\pi(K)$

```
function PI(K)
    G_1 ← a finite set of generators for K
    G_2 ← dual(G_1)                          ▷ a finite set of generators for K*
    G ← G_2 ⊗ G_1
    return mat(dual(vec(G)))
end function
```

4 Z-operators

We now move on to the **Z**-operators of Definition 9. Every **Z**-operator is the negation of some *cross-positive* operator—the class originally introduced by Schneider and Vidyasagar [21]. Elsner [8] and Tam [25] answered some early open questions about cross-positive operators. More work was done later by Gritzmann, Klee and Tam [13, 26]. Recently, Kuzma et al. [15] used cross-positive operators to answer an open question posed by Damm [7].

Many of the results hereafter would appear more natural (that is, without a minus sign) if stated in terms of cross-positive operators. However, the **Z**-operators—by way of **Z**-matrices—have historically received more attention, so we present our results in those terms. As before, we begin by pointing out that the set of all **Z**-operators on K forms a closed convex cone. Verification of the three criteria is straightforward.

Proposition 11. $\mathbf{Z}(K)$ *is a closed convex cone for any set K.*

If the ambient space V is nontrivial, then $\mathbf{Z}(K)$ contains the nontrivial subspace $\operatorname{span}(\{\operatorname{id}_V\})$ and is never proper in contrast with Theorem 3. It does however suffice to verify the **Z**-operator property on generating sets. This simplifies things greatly when K is polyhedral.

Proposition 12. *If $K = \operatorname{cone}(G_1)$ is closed in a Euclidean space and if $K^* = \operatorname{cone}(G_2)$, then $L \in \mathbf{Z}(K)$ if and only if*

$$\langle L(x), s\rangle \leq 0 \text{ for all } (x, s) \in C(K) \cap (G_1 \times G_2). \tag{2}$$

Proof. Clearly, if $L \in \mathbf{Z}(K)$, then L satisfies (2). So suppose that L satisfies (2) and let $(x, s) \in C(K)$. Since G_1 generates K and G_2 generates K^*, we can write $x = \sum \alpha_i x_i$ and $s = \sum \gamma_j s_j$. By expanding $\langle x, s\rangle = 0$ and noting that $\langle x_i, s_j\rangle \geq 0$, we see that each $(x_i, s_j) \in C(K)$. Linearity gives $\langle L(x), s\rangle \leq 0$. $\square$

As before, we want to find a generating set of $\mathbf{Z}(K)^*$ and use it to prove some results about $\mathbf{Z}(K)$. Recall from Section 3 that $\operatorname{cone}(K^* \otimes K) = \pi(K)^*$ for any closed convex cone K. We will take that fact for granted in this section. A similar set will generate $\mathbf{Z}(K)^*$. The following characterization is due to Bit-Shun Tam, and will simplify our remaining work.

Lemma 5. *If K is a closed convex cone in a Euclidean space V, then*

$$\operatorname{cone}(\{s \otimes x \mid (x, s) \in C(K)\}) = \pi(K)^* \cap \operatorname{span}(\{\mathrm{id}_V\})^{\perp}.$$

Proof. If $L = \sum s_i \otimes x_i$ with $(x_i, s_i) \in C(K)$, then $L \in \pi(K)^*$ by Theorem 1. Furthermore we have $\langle L, \mathrm{id}_V\rangle = \sum \langle x_i \otimes s_i, \mathrm{id}_V\rangle$, where each term satisfies $\langle x_i \otimes s_i, \mathrm{id}_V\rangle = \langle x_i, s_i\rangle = 0$. As a result we have $L \in \operatorname{span}(\{\mathrm{id}_V\})^{\perp}$.

On the other hand, if $L \in \pi(K)^* \cap \operatorname{span}(\{\mathrm{id}_V\})^{\perp}$, then Theorem 1 lets us write $L = \sum s_i \otimes x_i$ for $(x_i, s_i) \in K \times K^*$, and $\langle L, \mathrm{id}_V\rangle = 0$ expands to $\sum \langle x_i, s_i\rangle = 0$. Since each $\langle x_i, s_i\rangle$ is nonnegative, they all must be zero. Thus L is a conic combination of $s_i \otimes x_i$ terms with $(x_i, s_i) \in C(K)$. $\square$

Theorem 4. *If $K = \operatorname{cone}(G_1)$ is closed in a Euclidean space and if $K^* = \operatorname{cone}(G_2)$, then $\mathbf{Z}(K)^* = \operatorname{cone}(G)$ where G is the generating set*

$$G \coloneqq \{-s \otimes x \mid (x, s) \in C(K) \cap (G_1 \times G_2)\}.$$

Proof. We have $L \in \mathbf{Z}(K)$ if and only if $\langle -L(x), s\rangle = \langle L, -s \otimes x\rangle \geq 0$ for all $(x, s) \in C(K)$ by one property of the trace. Thus

$$L \in \mathbf{Z}(K) \iff L \in \operatorname{cone}(\{-s \otimes x \mid (x, s) \in C(K)\})^*.$$

Infer that the $\operatorname{cone}(\cdot)$ is closed from Lemma 5, and then take duals on both sides to obtain $\mathbf{Z}(K)^* = \operatorname{cone}(\{-s \otimes x \mid (x, s) \in C(K)\})$.

It remains only to show that $\operatorname{cone}(G) = \operatorname{cone}(\{-s \otimes x \mid (x, s) \in C(K)\})$. One inclusion is obvious, so let $L = \sum -s_i \otimes x_i$ where $(x_i, s_i) \in C(K)$. Expand each x_i and s_i in terms of G_1 and G_2 to obtain a sum of the form

$$L = \sum_{i,j,k} \alpha_{ij}\gamma_{ik}(-t_{ik} \otimes y_{ij}) \text{ where } \alpha_{ij}, \gamma_{ik} \geq 0, y_{ij} \in G_1, \text{ and } t_{ik} \in G_2. \tag{3}$$

Since $\langle L, \mathrm{id}_V\rangle = 0$ from Lemma 5, the linearity of the inner product and the fact that each $\langle y_{ij}, t_{ik}\rangle \geq 0$ together imply that all $\langle y_{ij}, t_{ik}\rangle = 0$, or that each $(y_{ij}, t_{ik}) \in C(K) \cap (G_1 \times G_2)$. Thus $L \in \operatorname{cone}(G)$. $\square$

The same result for proper cones follows from Lemma 2.2 in Gritzmann, Klee and Tam [13]. A few examples demonstrate how Theorem 4 can be used to find $\mathbf{Z}(K)^{*}$. In simple polyhedral cases, we are able to obtain $\mathbf{Z}(K)$ as well.

Example 8. If $K = \mathbb{R}^{n}_{+}$ in $V = \mathbb{R}^{n}$, then $C(K) = \{(e_i, e_j) \mid i \neq j\}$. Form $G := \left\{-e_j e_i^T \mid i \neq j\right\}$ to find that $\mathbf{Z}(K)^{*} = \operatorname{cone}(G)$ is the set of matrices whose diagonal entries are zero and whose off-diagonal entries are nonpositive. Its dual is the cone of $\mathbf{Z}$-matrices.

Example 9. If K is the half-space from Example 1, then Theorem 4 gives $\mathbf{Z}(K)^{*} = \mathbf{Z}(K)^{\perp} = \operatorname{span}\left(\left\{e_1 e_2^T\right\}\right)$. This result is verified by Proposition 12.

Corollary 2. *If K is a closed convex cone in a Euclidean space V and if ϕ and K_p are as in Lemma 1, then*

$$\mathbf{Z}(\phi(K)) = \begin{bmatrix} \mathcal{B}(V_1, V_1) & \mathcal{B}(V_2, V_1) & \mathcal{B}(V_3, V_1) \\ \{0\} & \mathbf{Z}(K_p) & \mathcal{B}(V_3, V_2) \\ \{0\} & \{0\} & \mathcal{B}(V_3, V_3) \end{bmatrix}.$$

Proof. Apply Theorem 4, Lemma 5, and Corollary 1 to $\phi(K)$ in the space $\phi(V)$:

$$-\mathbf{Z}(\phi(K))^{*} = \begin{bmatrix} \{0\} & \{0\} & \{0\} \\ \mathcal{B}(V_1, V_2) & \operatorname{cone}\left(K_p^{*} \otimes K_p\right) & \{0\} \\ \mathcal{B}(V_1, V_3) & \mathcal{B}(V_2, V_3) & \{0\} \end{bmatrix} \cap \operatorname{span}\left(\left\{\mathrm{id}_{\phi(V)}\right\}\right)^{\perp}.$$

Write $\operatorname{span}\left(\left\{\mathrm{id}_{\phi(V)}\right\}\right)$ in diagonal block form. The sets $\mathcal{B}(V_i, V_j)$ and $\{0\}$ are unaffected by the intersection, and $\operatorname{cone}\left(K_p^{*} \otimes K_p\right) \cap \operatorname{span}\left(\{\mathrm{id}_{V_2}\}\right)^{\perp} = -\mathbf{Z}(K_p)^{*}$ by Lemma 5 and Theorem 4. Take duals and negate both sides for the result. ☐

Theorem 5. *If K is a closed convex cone in a Euclidean space, then*

$$\dim(\mathbf{Z}(K)) = \dim(\pi(K)).$$

Proof. Compare Corollary 1 and Corollary 2 in view of Proposition 6. A priori we have $\dim(-\mathbf{Z}(K_p)) \geq \dim(\pi(K_p))$, but $\pi(K_p)$ is full-dimensional by Theorem 3, so the equality $\dim(\mathbf{Z}(K_p)) = \dim(\pi(K_p))$ follows. ☐

Theorem 6. *If K is a closed convex cone in a Euclidean space V, then $\mathbf{Z}(K)$ is polyhedral if and only if K is polyhedral.*

Proof. If K is polyhedral, then both K and K^{*} have finite generating sets by Proposition 1. Those generators combine via Theorem 4 to form a finite generating set of $\mathbf{Z}(K)^{*}$, showing that $\mathbf{Z}(K)$ is polyhedral (again by Proposition 1).

Let K_p be as in Lemma 1 and recall that K is polyhedral if and only if K_p is polyhedral by Proposition 5. If K is nonpolyhedral, then $\operatorname{Ext}(K_p)$ is infinite. To each $x \in \operatorname{Ext}(K_p)$ we can associate [21] a nonzero $s \in \operatorname{Ext}\left(K_p^{*}\right)$ with $\langle x, s\rangle = 0$. Tam proved [27] that the resulting $s \otimes x$ belongs to $\operatorname{Ext}\left(\pi(K_p)^{*}\right)$, and since it also belongs to $-\mathbf{Z}(K_p)^{*} \subseteq \pi(K_p)^{*}$, we must have $s \otimes x \in \operatorname{Ext}\left(\mathbf{Z}(K_p)^{*}\right)$. Thus $\operatorname{Ext}\left(\mathbf{Z}(K_p)^{*}\right)$ is infinite and $\mathbf{Z}(K_p)$ is nonpolyhedral by Proposition 1. Corollary 2 and Proposition 6 now show that $\mathbf{Z}(K)$ is nonpolyhedral. ☐

These results are corroborated by the polyhedral cones we have examined, all of which have polyhedral cones of **Z**-operators and satisfy $\dim(\mathbf{Z}(K)) = \dim(\pi(K))$.

Corollary 3. *If K is a closed convex cone in a Euclidean space, then $\mathbf{Z}(K)$ is polyhedral if and only if $\pi(K)$ is polyhedral.*

There are no simple characterizations of $\mathbf{Z}(K)$ for nonpolyhedral K. One sees an example in the work of Stern and Wolkowicz [23] who characterize the **Z**-operators on the Lorentz "ice cream" cone. We close this section with an algorithm, based on Theorem 4, to compute $\mathbf{Z}(K)$ for polyhedral K.

Algorithm 2 Compute generators of $\mathbf{Z}(K)$

Input: A polyhedral convex cone K
Output: A generating set of $\mathbf{Z}(K)$
  function $\mathrm{Z}(K)$
    $G_1 \leftarrow$ a finite set of generators for K
    $G_2 \leftarrow \mathrm{dual}(G_1)$ ▷ a finite set of generators for K^*
    $G \leftarrow \{-s \otimes x \mid x \in G_1, s \in G_2, \langle x, s \rangle = 0\}$
    return $\mathrm{mat}(\mathrm{dual}(\mathrm{vec}(G)))$
  end function

4.1 Composing Lyapunov-like operators

This is a convenient place to explain the observed behavior of Lyapunov-like operators on polyhedral convex cones. Recall that L is Lyapunov-like on K and we write $L \in \mathbf{LL}(K)$ if and only if both $\pm L \in \mathbf{Z}(K)$. From the following Theorem 2 of Gowda and Tao [12], it is easy to deduce that the composition of two Lyapunov-like operators on a proper polyhedral cone is itself Lyapunov-like.

Theorem 7. *If K is a proper polyhedral cone in a Euclidean space, then $L \in \mathbf{LL}(K)$ if and only if every $x \in \mathrm{Ext}(K)$ is an eigenvector of L.*

Lemma 6. *If K is a proper polyhedral cone in a Euclidean space V, then $\mathbf{LL}(K)$ is closed under composition, which is commutative.*

Proof. Theorem 7 shows that if L_1 and L_2 belong to $\mathbf{LL}(K)$, then $L_1 x = \lambda_1(x)\, x$ and $L_2 x = \lambda_2(x)\, x$ for all $x \in \mathrm{Ext}(K)$. As a result, $L_1 L_2 x = L_2 L_1 x = \lambda_2(x)\, \lambda_1(x)\, x$ for all $x \in \mathrm{Ext}(K)$. Apply Theorem 7 again to conclude that both $L_1 L_2$ and $L_2 L_1$ belong to $\mathbf{LL}(K)$. Thus, $\mathbf{LL}(K)$ is closed under composition. Global commutativity follows from the fact that $\mathrm{Ext}(K)$ spans V. □

One observes similar behavior when the cone in question is not proper. Lemma 6 has the following partial extension to a general polyhedral cone.

Theorem 8. *If K is a polyhedral convex cone in a Euclidean space V, then $\mathbf{LL}(K)$ is closed under composition.*

Proof. Proposition 6 and Corollary 2 extend in a natural way to $\mathbf{LL}(K)$—simply replace "$\mathbf{Z}$" by "$\mathbf{LL}$" everywhere. Keeping the block-upper-triangular representation of $\mathbf{LL}(\phi(K))$ in mind, and by applying Lemma 6 to $\mathbf{LL}(K_p)$, one readily shows that $\mathbf{LL}(\phi(K))$ is closed under composition. It follows that $\mathbf{LL}(K) = \phi^{-1}\mathbf{LL}(\phi(K))\,\phi$ is also closed under composition. □

Note that in going from proper to closed and convex, we have lost commutativity. One easily finds nonpolyhedral proper cones whose Lyapunov-like operators are not closed under composition. The Lyapunov-like operators on the symmetric positive semidefinite cone [7] are the Lyapunov transformations $L_A(X) := AX + XA^T$, and they are not closed under composition. Nor is the conclusion of Theorem 8 exclusive to polyhedral cones: most cones [13] have $\mathbf{LL}(K) = \operatorname{span}(\{\mathrm{id}_V\})$, trivially closed under composition.

5 The exponential connection

Finally we exhibit an explicit connection between positive and $\mathbf{Z}$-operators. As discovered by Schneider and Vidyasagar [21] and Elsner [8], it applies to proper cones. We restate their theorem in slightly more general language.

Theorem 9. *If K is a proper cone in a Euclidean space V and if $L \in \mathcal{B}(V)$, then $L \in \mathbf{Z}(K)$ if and only if $e^{-tL} \in \pi(K)$ for all $t \geq 0$.*

This theorem has been used effectively. Elsner [8] equates exponentially-positive, resolvent-positive, essentially-positive, cross-positive, and quasimonotone operators. Damm [7] shows that Lyapunov-like operators on the positive-semidefinite cone are the familiar Lyapunov transformations from dynamical systems. Gowda, Tao, and Orlitzky [12, 16] characterize the Lie algebra of the automorphism group of a closed convex cone.

Lemma 7. *If K is a subset of a Euclidean space V and if $L \in \mathcal{B}(V)$ with $e^{-tL} \in \pi(K)$ for all $t \geq 0$, then $L \in \mathbf{Z}(K)$.*

Proof. Let $e^{-tL} \in \pi(K)$ for all $t \geq 0$, and take any $(x, s) \in C(K)$. We show that $\langle L(x), s\rangle \leq 0$ and it follows that $L \in \mathbf{Z}(K)$. Since $e^{-tL}(x) \in K$,

$$\frac{1}{t}\left\langle \left[e^{-tL} - \mathrm{id}_V\right](x), s\right\rangle = \frac{1}{t}\left\langle e^{-tL}(x), s\right\rangle \geq 0 \text{ for all } t > 0.$$

Take the limit as $t \to 0$ to find $\langle L(x), s\rangle \leq 0$. □

To prove the converse of Lemma 7, we will ultimately rely on Theorem 9 for proper cones. To do that we will appeal to the decomposition in Section 2.4.

Theorem 10. *If K is a closed convex cone in a Euclidean space V and if $L \in \mathcal{B}(V)$, then $L \in \mathbf{Z}(K) \iff e^{-tL} \in \pi(K)$ for all $t \geq 0$.*

Proof. One implication was already shown in Lemma 7. If we let ϕ and K_p be as in Lemma 1, then the converse of Lemma 7 holds for $\phi(K)$. To see why, suppose that $L = \left[L_{ij}\right] \in \mathbf{Z}(\phi(K))$ has the block form of Corollary 2. Then $L_{22} \in \mathbf{Z}(K_p)$, and from Theorem 9 we obtain $e^{-tL_{22}} \in \pi(K_p)$ for all $t \geq 0$. Now, by appealing to the block-upper-triangular form of L, exponentiate directly:

$$e^{-tL} = \sum_{n=0}^{\infty} \frac{1}{n!}(-tL)^n = \begin{bmatrix} e^{-tL_{11}} & A & B \\ 0 & e^{-tL_{22}} & D \\ 0 & 0 & e^{-tL_{33}} \end{bmatrix}.$$

We are not interested in the precise form of A, B, and D. Apply Corollary 1 to conclude that $e^{-tL} \in \pi(\phi(K))$ for all $t \geq 0$, and use Proposition 6 to eliminate ϕ from the result. ☐

A similar result appears in Hilgert, Hofmann, and Lawson [14]. The first two items of their Theorem III.1.9 state that $L \in \mathbf{Z}(K)$ if and only if $e^{-tL} \in \pi(K)$ for all $t \geq 0$. However, the remaining items suggest hidden assumptions, and its proof relies on another Theorem I.5.27 where the cone is solid. Nevertheless, their Theorem I.5.17 seems to provide the machinery needed to prove the result.

All of our previous examples corroborate Theorem 10. We provide an application to dynamical systems.

Example 10. The system $x'(t) = -L(x(t))$ has solution $x(t) = e^{-tL}(x(0))$. If $L \in \mathbf{Z}(K)$ for some closed convex cone K, then Theorem 10 shows that $e^{-tL} \in \pi(K)$ for all $t \geq 0$. Therefore $x(t)$ remains in K for $t > 0$ if $x(0) \in K$.

Theorem 4 of Orlitzky [16] now follows as a corollary.

Corollary 4. *If K is a closed convex cone in a Euclidean space, then $\mathbf{LL}(K)$ is the Lie algebra of the automorphism group of K.*

Proof. Apply Theorem 10 to both $\pm L \in \mathbf{Z}(K)$. ☐

When $K = V = \mathbb{R}^n$, this witnesses the well-known fact that the $n \times n$ real matrices are the Lie algebra of the general linear group of degree n over $\mathbb{R}$.

6 Decomposing Z-operators

Any $L \in \mathbf{Z}\left(\mathbb{R}^n_+\right)$ is of the form $L = \lambda I - N$ where $\lambda \in \mathbb{R}$ and $N \in \pi\left(\mathbb{R}^n_+\right)$ is a nonnegative matrix [4]. Schneider and Vidyasagar [21] show that a similar decomposition exists for any proper polyhedral cone: if K is proper and polyhedral in V, then $\mathbf{Z}(K) = \operatorname{span}(\{\operatorname{id}_V\}) - \pi(K)$. The authors leave open the question of when such a decomposition exists. The answer is "almost never" [13], but we do always have $\mathbf{Z}(K) = \operatorname{cl}(\operatorname{span}(\{\operatorname{id}_V\}) - \pi(K))$ if we take the closure [21].

An **M**-matrix is a **Z**-matrix all of whose eigenvalues have nonnegative real parts. Early attempts to generalize **M**-matrices to a proper cone K in V involved operators of the form $\operatorname{span}(\{\operatorname{id}_V\}) - \pi(K)$, and operators having that form are called *K-regular* [22]. A K-regular matrix whose eigenvalues have nonnegative real parts is called a K-*general* $\boldsymbol{M}$-*matrix*. But recall that K-regularity is not necessarily equivalent to membership in $\mathbf{Z}(K)$ when K is non-polyhedral [13]. Using the geometric notion of subtangentiality, Stern and Tsatsomeros [22] remedy that situation by introducing K-*extended* $\boldsymbol{M}$-*matrices* defined in terms of the exponential positivity that (by Theorem 10, for example) characterizes $\mathbf{Z}(K)$. It is shown that every K-extended $\mathbf{M}$-matrix is the limit of K-general $\mathbf{M}$-matrices.

From Definition 10 it should be obvious that $\operatorname{span}(\{\mathrm{id}_V\}) \subseteq \mathbf{LL}(K) \subseteq \mathbf{Z}(K)$ for any closed convex cone K. It therefore makes sense to investigate when $\mathbf{LL}(K) - \pi(K) = \mathbf{Z}(K)$. Damm [7] asks if this is true for the cone of symmetric or Hermitian positive-semidefinite matrices (either real or complex). Kuzma et al. [15] provide an answer, constructing a counterexample when the matrices are larger than 2×2. In the process, the authors show that if K is the cone of squares in a simple Euclidean Jordan algebra V, then $\mathbf{Z}(K) = \mathbf{LL}(K) - \pi(K)$ if and only if the rank of V is 2 or less.

With Theorem 10 at our disposal, we can prove an analogue of the result obtained by Schneider and Vidyasagar.

Theorem 11. *If K is a closed convex cone in a Euclidean space V, then $\mathbf{Z}(K) = \operatorname{cl}(\operatorname{span}(\{\mathrm{id}_V\}) - \pi(K))$.*

Proof. We have $\operatorname{span}(\{\mathrm{id}_V\}) - \pi(K) \subseteq \mathbf{Z}(K)$ from their definitions. Thus $\operatorname{cl}(\operatorname{span}(\{\mathrm{id}_V\}) - \pi(K)) \subseteq \operatorname{cl}(\mathbf{Z}(K)) = \mathbf{Z}(K)$ because $\mathbf{Z}(K)$ is closed.

If $L \in \mathbf{Z}(K)$, then $e^{-tL} \in \pi(K)$ for all $t \geq 0$ by Theorem 10. The function $f(t) := \left(\mathrm{id}_V - e^{-tL}\right)/t$ converges to L as $t > 0$ approaches zero, and $f(t) \in \operatorname{span}(\{\mathrm{id}_V\}) - \pi(K)$ for all $t > 0$. Thus $L \in \operatorname{cl}(\operatorname{span}(\{\mathrm{id}_V\}) - \pi(K))$. ☐

To demonstrate the power of Theorem 11, we will use it to construct new proofs of Theorem 4, Theorem 5, and half of Theorem 6.

Corollary 5. *If K is a closed convex cone in a Euclidean space V, then $\mathbf{Z}(K)^* = \operatorname{cone}(\{-s \otimes x \mid (x, s) \in C(K)\})$.*

Proof. Take duals in Theorem 11 and apply Proposition 2 to find $\mathbf{Z}(K)^* = \operatorname{span}(\{\mathrm{id}_V\})^\perp \cap \left(-\pi(K)^*\right)$. Now consult Lemma 5. ☐

Corollary 6. *If K is a closed convex cone in a Euclidean space V, then $\dim(\mathbf{Z}(K)) = \dim(\pi(K))$.*

Proof. Use Theorem 11 to obtain $\dim(\mathbf{Z}(K)) = \dim(\operatorname{span}(\{\mathrm{id}_V\}) - \pi(K))$ which is defined to be $\dim(\operatorname{span}(\operatorname{span}(\{\mathrm{id}_V\}) - \pi(K)))$. But $\mathrm{id}_V \in \pi(K)$, so

$$\dim(\mathbf{Z}(K)) = \dim(\operatorname{span}(\pi(K))) =: \dim(\pi(K)). \qquad \square$$

Corollary 7. *If K is a polyhedral convex cone in a Euclidean space, then $\mathbf{Z}(K)$ is polyhedral.*

Proof. If K is polyhedral, then $\pi(K)$ is polyhedral. It therefore follows that $\operatorname{span}(\{\mathrm{id}_V\}) - \pi(K)$, being the sum of two polyhedral cones, is both polyhedral and closed. Thus $\mathbf{Z}(K) = \operatorname{cl}(\operatorname{span}(\{\mathrm{id}_V\}) - \pi(K)) = \operatorname{span}(\{\mathrm{id}_V\}) - \pi(K)$. ☐

The converse of Corollary 7 seems more elusive.

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References

[1] Charalambos D. Aliprantis and Owen Burkinshaw. *Positive Operators*. Springer, Dordrecht, The Netherlands, 2006. ISBN 9781402050077, doi:10.1007/978-1-4020-5008-4.

[2] Adi Ben-Israel. *Linear equations and inequalities on finite dimensional, real or complex, vector spaces: A unified theory*. Journal of Mathematical Analysis and Applications, 27(2):367–389, 1969, doi:10.1016/0022-247X(69)90054-7.

[3] Aharon Ben-Tal and Arkadi Nemirovski. *Lectures on Modern Convex Optimization*. MPS/SIAM Series on Optimization. SIAM, Philadelphia, 2001. ISBN 9780898714913, doi:10.1137/1.9780898718829.

[4] Abraham Berman and Robert J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences*. SIAM, Philadelphia, 1994. ISBN 9780898713213, doi:10.1137/1.9781611971262.

[5] Jonathan M. Borwein and Michael. A.H. Dempster. *The linear order complementarity problem*. Mathematics of Operations Research, 14:534–558, 1989, doi:10.1287/moor.14.3.534.

[6] Colin W. Cryer and Michael A.H. Dempster. *Equivalence of linear complementarity problems and linear programs in vector lattice Hilbert spaces*. SIAM Journal on Control and Optimization, 18:76–90, 1980, doi:10.1137/0318005.

[7] Tobias Damm. *Positive groups on $\mathcal{H}^n$ are completely positive*. Linear Algebra and its Applications, 393:127–137, 2004, doi:10.1016/j.laa.2003.12.045.

[8] Ludwig Elsner. *Quasimonotonie und ungleichungen in halbgeordneten räumen*. Linear Algebra and its Applications, 8:249–261, 1974, doi:10.1016/0024-3795(74)90070-6.

[9] Muddappa Seetharama Gowda and Gomatam Ravindran. *On the game-theoretic value of a linear transformation relative to a self-dual cone*. Linear Algebra and its Applications, 469:440–463, 2015, doi:10.1016/j.laa.2014.11.032.

[10] Muddappa Seetharama Gowda, Roman Sznajder, and Jiyuan Tao. *The automorphism group of a completely positive cone and its Lie algebra*. Linear Algebra and its Applications, 438:3862–3871, 2013, doi:10.1016/j.laa.2011.10.006.

[11] Muddappa Seetharama Gowda and Jiyuan Tao. *Z-transformations on proper and symmetric cones*. Mathematical Programming, 117(1–2):195–222, 2009, doi:10.1007/s10107-007-0159-8.

[12] Muddappa Seetharama Gowda and Jiyuan Tao. *On the bilinearity rank of a proper cone and Lyapunov-like transformations*. Mathematical Programming, 147:155–170, 2014, doi:10.1007/s10107-013-0715-3.

[13] Peter Gritzmann, Victor Klee, and Bit-Shun Tam. *Cross-positive matrices revisited*. Linear Algebra and its Applications, 223–224:285–305, 1995, doi:10.1016/0024-3795(93)00364-6.

[14] Joachim Hilgert, Karl Heinrich Hofmann, and Jimmie D. Lawson. *Lie groups, convex cones and semigroups*. Oxford Mathematical Monographs. Clarendon Press, Oxford, 1989. ISBN 9780198535690.

[15] Bojan Kuzma, Matjaž Omladič, Klemen Šivic, and Josef Teichmann. *Exotic one-parameter semigroups of endomorphisms of a symmetric cone*. Linear Algebra and its Applications, 477:42–75, 2015, doi:10.1016/j.laa.2015.03.006.

[16] Michael Orlitzky. *The Lyapunov rank of an improper cone*. Optimization Methods and Software, 32(1):109–125, 2017, doi:10.1080/10556788.2016.1202246.

[17] Ralph Tyrrell Rockafellar. *Convex Analysis*. Princeton University Press, Princeton, 1970. ISBN 9780691015866.

[18] Steven Roman. *Advanced Linear Algebra*, vol. 135 of *Graduate Texts in Mathematics*. Springer Science+Business Media, New York, third ed., 2008. ISBN 9780387728285, doi:10.1007/978-0-387-72831-5.

[19] Gábor Rudolf, Nilay Noyan, Dávid Papp, and Farid Alizadeh. *Bilinear optimality constraints for the cone of positive polynomials*. Mathematical Programming, 129:5–31, 2011, doi:10.1007/s10107-011-0458-y.

[20] Hans Schneider. *Positive operators and an inertia theorem*. Numerische Mathematik, 7:11–17, 1965.

[21] Hans Schneider and Mathukumalli Vidyasagar. *Cross-positive matrices*. SIAM Journal on Numerical Analysis, 7:508–519, 1970, doi:10.1137/0707041.

[22] Ronald. J. Stern and Michael Tsatsomeros. *Extended M-matrices and subtangentiality*. Linear Algebra and its Applications, 97:1–11, 1987, doi:10.1016/0024-3795(87)90134-0.

[23] Ronald. J. Stern and Henry Wolkowicz. *Exponential nonnegativity on the ice cream cone*. SIAM Journal on Matrix Analysis and Applications, 12:160–165, 1991, doi:10.1137/0612012.

[24] Josef Stoer and Christoph Witzgall. *Convexity and Optimization in Finite Dimensions I*, vol. 163 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, New York, 1970. ISBN 9783642462184, doi:10.1007/978-3-642-46216-0.

[25] Bit-Shun Tam. *A note on cross-positive matrices*. Linear Algebra and its Applications, 12:7–9, 1975, doi:10.1016/0024-3795(75)90122-6.

[26] Bit-Shun Tam. *Some results on cross-positive matrices*. Linear Algebra and its Applications, 15:173–176, 1976, doi:10.1016/0024-3795(76)90014-8.

[27] Bit-Shun Tam. *Some results of polyhedral cones and simplicial cones*. Linear and Multilinear Algebra, 4(4):281–284, 1977, doi:10.1080/03081087708817164.

[28] The Sage Developers. *SageMath, the Sage Mathematics Software System*, 2017. URL http://www.sagemath.org/.

[29] Günter M. Ziegler. *Lectures on Polytopes*, vol. 152 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. ISBN 9780387943657, doi:10.1007/978-1-4613-8431-1.