Proscribed normal decompositions of Euclidean Jordan algebras

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Abstract

Normal decomposition systems unify many results from convex matrix analysis regarding functions that are invariant with respect to a group of transformations—particularly those matrix functions that are unitarily-invariant and the affiliated permutation-invariant "spectral functions" that depend only on eigenvalues. Spectral functions extend in a natural way to Euclidean Jordan algebras, and several authors have studied the problem of making a Euclidean Jordan algebra into a normal decomposition system. In particular it is known to be possible with respect to the "eigenvalues of" map when the algebra is essentially-simple. We show the converse, that essential-simplicity is essential to that process.

Keywords: Normal decomposition system, Eaton triple, Spectral function, Group majorization, Euclidean Jordan algebra

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1 Notation

All spaces we encounter will be finite-dimensional real inner-product spaces with the generic $\langle x, y \rangle$ denoting the inner product of x and y. If V is such a space, then $\mathcal{B}(V)$ is the set of all linear operators on V and $\operatorname{Aut}(X)$ is the linear automorphism group of $X \subseteq V$. The inner product on V induces a norm, and $\operatorname{Isom}(V)$ is the group of linear isometries on V under composition. Explicitly,

$$\operatorname{Aut}(X) := \left\{ L \in \mathcal{B}(V) \;\middle|\; L^{-1} \text{ exists and } L(X) = X \right\},$$
$$\operatorname{Isom}(V) := \{ L \in \operatorname{Aut}(V) \mid \|L(x)\| = \|x\| \text{ for all } x \text{ in } V \}.$$

The group $\operatorname{Isom}(V)$ is endowed with the natural topology [16] so that L_n approaches L in $\operatorname{Isom}(V)$ if and only if $L_n(x)$ approaches $L(x)$ for all $x \in V$. The adjoint of any $L \in \mathcal{B}(V)$ is denoted by L^*, and thus $L^* = L^{-1}$ when $L \in \operatorname{Isom}(V)$. We will refer occasionally to the following classes of matrices:

$\mathcal{H}^n$ – complex Hermitian n-by-n matrices,

Σ^n – real n-by-n permutation matrices,

Γ^n – real diagonal n-by-n matrices with ± 1 entries.

Closed convex cones play an important part in both normal decomposition systems and Euclidean Jordan algebras. A nonempty subset K of V is a cone if $\alpha K \subseteq K$ for all $\alpha \geq 0$. A *closed convex cone* is a cone that is closed and convex as a subset of V. The *dual cone* of K is itself a closed convex cone, denoted by

$$K^* \coloneqq \{y \in V \mid \langle x, y \rangle \geq 0 \text{ for all } x \in K\}.$$

If $K = K^*$, then K is *self-dual*. For example, the nonnegative orthant $\mathbb{R}^n_+$ is a self-dual closed convex cone in $\mathbb{R}^n$.

2 Normal decomposition systems

A *spectral function* is a real-valued function of a real or complex Hermitian matrix that depends only on the eigenvalues of its argument. Friedland [5] is due credit for the appellation, but the idea goes back to Davis [2]. Lewis [15] adopted the subject in 1996, to which pertains the following summary.

If $F : \mathcal{H}^n \to \mathbb{R}$ is a spectral function, then $F(X) = F(UXU^*)$ for all $X \in \mathcal{H}^n$ and all unitary $U \in \mathbb{C}^{n \times n}$. Let $\lambda^{\downarrow} : \mathcal{H}^n \to \mathbb{R}^n$ be the function that takes a matrix to the vector of its eigenvalues, arranged in nonincreasing order. Since any Hermitian matrix can be diagonalized by unitary matrices, we must have $F(X) = F\left(\operatorname{diag}\left(\lambda^{\downarrow}(X)\right)\right)$ for all X. We therefore need only concern ourselves with the action of F on diagonal matrices whose entries from top-left to bottom-right are arranged in nonincreasing order.

To every such F there corresponds an $f : \mathbb{R}^n \to \mathbb{R}$ such that $f\left(\lambda^{\downarrow}(X)\right) = F\left(\operatorname{diag}\left(\lambda^{\downarrow}(X)\right)\right) = F(X)$ and vice-versa. (It is less obvious that convexity is preserved in both directions [2].) For that reason, functions of the form $f \circ \lambda^{\downarrow}$ are also referred to as spectral functions. This charade gains us the privilege of working in $\mathbb{R}^n$ as opposed to the larger, more complicated $\mathcal{H}^n$. And since the argument to f will be arranged in nonincreasing order, we may further restrict the definition of f to the so-called "downward-monotonic cone" denoted by $(\mathbb{R}^n)^{\downarrow}$ whose elements' components have been thusly arranged. This last step reveals that much of the redundancy in a spectral function is eliminated through the rearrangement map $x \mapsto x^{\downarrow}$ that exhibits several important properties:

1. If P is any permutation matrix, then $(P(x))^{\downarrow} = x^{\downarrow}$ for all x.
2. For any $x \in \mathbb{R}^n$ there is a permutation matrix P with $P\left(x^{\downarrow}\right) = x$.
3. The inequality $\langle x, y \rangle \leq \left\langle x^{\downarrow}, y^{\downarrow} \right\rangle$ holds for all $x, y \in \mathbb{R}^n$.

In general, if $\gamma : V \to V$ and if G is some set of functions on V, then we say that γ is *G-invariant* if $\gamma(g(x)) = \gamma(x)$ for all $x \in V$ and all $g \in G$. So the map $x \mapsto x^{\downarrow}$ is said to be Σ^n-invariant, or simply "permutation-invariant." These properties are what Lewis axiomatized into a normal decomposition system.

Definition 1 (Normal decomposition system, Lewis [16], Definition 2.1)**.** A normal decomposition system (V, G, γ) consists of

1. A finite-dimensional real inner-product space V.
2. A closed subgroup G of the isometry group $\operatorname{Isom}(V)$.
3. A map $\gamma : V \to V$ satisfying the following:
   (a) G-invariance: $\gamma(g(x)) = \gamma(x)$ for all $x \in V$ and all $g \in G$.
   (b) For all $x \in V$ there exists some $g \in G$ with $g(\gamma(x)) = x$.
   (c) The inequality $\langle x, y\rangle \leq \langle \gamma(x), \gamma(y)\rangle$ holds for all $x, y \in V$.

Following that definition, Lewis remarks that the map γ is idempotent and preserves norms. His Theorem 2.4 later shows that γ is positive-homogeneous and Lipschitz continuous, and that $\gamma(V)$ is a closed convex cone in V. Fixing $c \in V$, the applicability of Definition 1 to optimization problems of the form

$$\text{maximize } f(x) \coloneqq \langle c, x\rangle \text{ subject to } x \in V$$

is visible in the following result.

Theorem 1 (Lewis [16], Theorem 2.2 and Proposition 2.3)**.** *If (V, G, γ) is a normal decomposition system and if $x, y \in V$, then*

$$\langle \gamma(x), \gamma(y)\rangle = \max\left(\{\langle x, g(y)\rangle \mid g \in G\}\right),$$

and $\langle x, y\rangle = \langle \gamma(x), \gamma(y)\rangle$ if and only if there exists some $g \in G$ with $x = g(\gamma(x))$ and $y = g(\gamma(y))$.

Normal decomposition systems "correspond exactly" to the group-induced cone preorders that arise in the theory of Eaton triples [17, 27]. If we denote the convex hull of X by $\operatorname{conv}(X)$, then classical majorization [19] is a preordering on $V = \mathbb{R}^n$ induced by the permutation group $G = \Sigma^n$:

$$x \preccurlyeq_G y \iff x \in \operatorname{conv}(G(y)).$$

This is the setting of Schur convexity, and the idea extends mechanically to the concept of group majorization, wherein the permutation group is replaced by some other closed subgroup G of $\operatorname{Isom}(V)$. A *group-induced cone preorder* is then a group majorization that happens to come from a closed convex cone. Adopting Niezgoda's Definition 2.2, that closed convex cone is precisely $\gamma(V)$, but one should beware that there is a specific dual-cone operation involved [24]. We recall our motivating example, which is a conglomerate of Examples 7.1, 7.3, and 7.4 of Lewis [16] and Example 2.2 of Niezgoda [24].

Example 1. If $V = \mathbb{R}^n$, if $G = \Sigma^n$, and if $\gamma = x \mapsto x^{\downarrow}$, then (V, G, γ) forms a normal decomposition system by the discussion preceding Definition 1. The optimality condition of Theorem 1 in this setting is classical and dates back to Theorem 368 of Hardy, Littlewood, and Pólya [10], which says that $\langle x, y\rangle = \left\langle x^{\downarrow}, y^{\downarrow}\right\rangle$ if and only if x and y are in a "similar order." The group-induced cone preordering corresponding to this normal decomposition system is the majorization ordering [19, 24], and its cone is the downward-monotonic cone $\gamma(V) = (\mathbb{R}^n)^{\downarrow}$ whose elements' components are arranged in nonincreasing order. Its dual in $\mathbb{R}^n$ is the *Schur cone* $\Phi_n \coloneqq \left((\mathbb{R}^n)^{\downarrow}\right)^*$ that induces the majorization ordering via $x \preccurlyeq_G y \iff \gamma(y) - \gamma(x) \in \Phi_n$. The generators of the Schur cone are given explicitly, for example, in Example 7.3 of Iusem and Seeger [12],

$$\Phi_n = \operatorname{cone}\left(\{e_i - e_{i+1} \mid i = 1, 2, \ldots, n-1\}\right),$$

where $\{e_1, e_2, \ldots, e_n\}$ is the standard basis in $\mathbb{R}^n$ and $\operatorname{cone}(\cdot)$ is the conic hull.

If we transfer the properties of γ to $\lambda^{\downarrow}$, then the spectral-function correspondence $F(X) = f\left(\lambda^{\downarrow}(X)\right)$ and Theorem 1 can be interpreted for Hermitian matrices X and Y to mean that $\operatorname{trace}(XY) \leq \left\langle \lambda^{\downarrow}(X), \lambda^{\downarrow}(Y) \right\rangle$ with equality if and only if X and Y are simultaneously-diagonalizable. If we include the equality condition, this result is due to Theobald [28] and is summarized in Theorem 2.2 of Lewis [15].

Motivated by the simultaneous-diagonalizability in the previous example, two elements $x, y \in V$ are said to *commute* if they satisfy the condition for equality in Theorem 1. This terminology will only be needed by the reader who chooses to traverse the bibliography. The next example is an amalgamation of Examples 7.2, 7.5 and 7.6 of Lewis [16] and Example 2.3 of Niezgoda [24].

Example 2. Recall the reflection group Γ^n consisting of real n-by-n diagonal matrices with ± 1 entries. If $V = \mathbb{R}^n$, if $G = \{RP \mid R \in \Gamma^n, P \in \Sigma^n\}$ and if $\gamma = x \mapsto |x|^{\downarrow}$, then (V, G, γ) is a normal decomposition system. Grove and Benson [8] prove in Section 5.3 that G is the group generated by $\Sigma^n \cup \Gamma^n$. The fact that it is topologically closed follows from its finitude.

First we show that γ is G-invariant. Let $g = RP \in G$; then $\gamma(g(x)) = |R(P(x))|^{\downarrow}$. Now $|R(y)|$ is clearly equal to $|y|$, and it's similarly easy to see that $|P(y)| = P(|y|)$. Therefore $\gamma(g(x)) = (P(|x|))^{\downarrow}$, and we can use the fact from the previous example that $x \mapsto x^{\downarrow}$ is permutation-invariant to conclude that $\gamma(g(x)) = |x|^{\downarrow} = \gamma(x)$.

Second, we must show that for all $x \in \mathbb{R}^n$, there exists some $g \in G$ with $x = g(\gamma(x))$. Let $x = (x_1, x_2, \ldots, x_n)^T$ be given. Clearly we can write

$$x = (s_1 |x_1|, s_2 |x_2|, \ldots, s_n |x_n|)^T$$

for $s_i \in \{-1, 1\}$. Thus, $x = R(|x|)$ for $R \coloneqq \operatorname{diag}(s_1, s_2, \ldots, s_n) \in \Gamma^n$. Now from the previous example, there exists some $P \in \Sigma^n$ such that $P\left(|x|^{\downarrow}\right) = |x|$. Substituting gives $x = R\left(P\left(|x|^{\downarrow}\right)\right)$, and if we let $g = RP$, then $x = g(\gamma(x))$.

Finally, we must show that for all $x, y \in \mathbb{R}^n$ we have $\langle x, y \rangle \leq \langle \gamma(x), \gamma(y) \rangle$. It should be obvious from the definition that $\langle x, y \rangle \leq \langle |x|, |y| \rangle$. Thus, once more by the previous example, it follows that

$$\langle x, y \rangle \leq \langle |x|, |y| \rangle \leq \left\langle |x|^{\downarrow}, |y|^{\downarrow} \right\rangle = \langle \gamma(x), \gamma(y) \rangle.$$

We conclude that (V, G, γ) is a normal decomposition system. The group-induced cone preordering corresponding to this normal decomposition system is the absolute *weak majorization* preordering [19, 24],

$$x \preccurlyeq_G y \iff \sum_{i=1}^{j} |x|_i^{\downarrow} \leq \sum_{i=1}^{j} |y|_i^{\downarrow} \text{ for all } 1 \leq j \leq n.$$

The cone associated with this system is $\gamma(\mathbb{R}^n) = (\mathbb{R}^n)^{\downarrow} \cap \mathbb{R}^n_+$. Recall the Schur cone Φ_n from Example 1. The dual of $\gamma(\mathbb{R}^n)$ can be computed directly using the fact that all cones involved are polyhedral and therefore closed:

$$(\gamma(\mathbb{R}^n))^* = \left((\mathbb{R}^n)^{\downarrow} \cap \mathbb{R}^n_+\right)^* = \operatorname{cl}\left(\left((\mathbb{R}^n)^{\downarrow}\right)^* + \left(\mathbb{R}^n_+\right)^*\right) = \Phi_n + \mathbb{R}^n_+.$$

A generating set for $\Phi_n + \mathbb{R}^n_+$ consists of the union of the generators for Φ_n and the standard basis, which is a generating set for $\mathbb{R}^n_+$. The absolute weak majorization preorder is $x \preccurlyeq_G y \iff \gamma(y) - \gamma(x) \in \Phi_n + \mathbb{R}^n_+$.

Our first example of a normal decomposition system was borne of matrix functions that depend only on the eigenvalues of their arguments. This example is motivated by functions that depend only on *singular values*. If σ denotes "the singular values of," then these functions F will satisfy $F(X) = F(UXV)$ for all unitary U and V, and correspond to functions $f \circ \sigma^{\downarrow}$ via $F(X) = F\left(\operatorname{diag}\left(\sigma^{\downarrow}(X)\right)\right) = f\left(\sigma^{\downarrow}(X)\right)$. If we restrict our attention to norms, then said functions are essentially the unitarily-invariant norms described in Section 3.5 of Horn and Johnson [11]. The function $\sigma^{\downarrow}$ is invariant under the group G, and that allows us to study the unitarily-invariant norms by studying the functions $f \circ \sigma^{\downarrow}$ on the closed convex cone $\gamma(\mathbb{R}^n) = (\mathbb{R}^n)^{\downarrow} \cap \mathbb{R}^n_+$.

After transferring the properties of γ to $\sigma^{\downarrow}$, the correspondence $F(X) = f\left(\sigma^{\downarrow}(X)\right)$ and Theorem 1 can be interpreted for any matrices X and Y to be "von Neumann's Lemma," that $|\operatorname{trace}(XY)| \leq \left\langle \sigma^{\downarrow}(X), \sigma^{\downarrow}(Y)\right\rangle$ with equality if and only if X and Y have simultaneous singular-value decompositions [21, 22]. The equality condition is discussed in Remark 1.2 of de Sá [3].

One consequence of Item 3a in a normal decomposition system is that γ sends the entire G-orbit of any x to $\gamma(x)$. Conversely, by Item 3b, if $\gamma(y) = \gamma(x)$, then $y \in G(x)$. Thus γ partitions the ambient space V into equivalence classes of G-orbits that we will denote by

$$[x]_{\gamma} \coloneqq \{y \in V \mid \gamma(x) = \gamma(y)\}.$$

The assemblage of all such classes is written V/γ. So, for example, one has $[x]_{\gamma} \in V/\gamma$. We will use this notation freely with functions other than γ.

Proposition 1. *If V is a vector space, if $G \subseteq \mathcal{B}(V)$ is a group, and if $\gamma : V \to V$, then γ is G-invariant if and only if $g\left([x]_{\gamma}\right) = [x]_{\gamma}$ for all $g \in G$ and $x \in V$.*

Proof. Suppose that the second condition holds, and let $x \in V$ be arbitrary. Note that, by assumption, we have $g(x) \in [x]_{\gamma}$ for all $g \in G$. It follows that $\gamma(g(x)) = \gamma(x)$ for all $g \in G$, showing that γ is G-invariant.

For the other implication, suppose that γ is G-invariant. Let $x \in V$, $g \in G$, and $y \in [x]_\gamma$ be otherwise arbitrary. To see that $g\left([x]_\gamma\right) \subseteq [x]_\gamma$, apply γ to $g(y)$, and use G-invariance: $\gamma(g(y)) = \gamma(y) = \gamma(x)$. Thus, $g(y) \in [x]_\gamma$. For the other inclusion, recall that G is a group, and let $z := g^{-1}(y)$. Then $y = g(z)$, and by G-invariance, $\gamma(z) = \gamma\left(g^{-1}(y)\right) = \gamma(y) = \gamma(x)$ implying that $z \in [x]_\gamma$. It follows that $y \in g\left([x]_\gamma\right)$. ☐

Corollary 1. *If (V, G, γ) is a normal decomposition system, then*

$$g\left([x]_\gamma\right) = [x]_\gamma \text{ for all } x \in V \text{ and all } g \in G,$$

and in particular $G \subseteq \operatorname{Aut}(X)$ for every $X \in V/\gamma$.

A major theme of normal decomposition systems is that the ambient space decomposes as $V = G(\gamma(V))$. However, this isn't quite what the axioms require. Item 3b states that any point x must not only decompose into $g(y)$ for some $g \in G$ and $y \in \gamma(V)$, but that, in addition, we must be able to choose $y = \gamma(x)$. A priori this is stronger than the requirement that $V = G(\gamma(V))$, but our next result reveals some circumstances that make the two conditions equivalent.

Proposition 2. *If V is a vector space, if $G \subseteq \mathcal{B}(V)$ is a group, and if $\gamma : V \to V$ is G-invariant with $\gamma^2 = \gamma$, then the following are equivalent:*

1. *For all $x \in V$, there exists some $g \in G$ with $g(\gamma(x)) = x$.*
2. *G acts transitively on each equivalence class in V/γ.*
3. *$G(\gamma(V)) = V$.*

Proof. Suppose that the first condition holds and let y, z belong to any $[x]_\gamma$. Then there exist g_y and g_z in G such that $g_y^{-1}(y) = \gamma(y) = \gamma(x) = \gamma(z) = g_z^{-1}(z)$, and multiplying both sides by either g_y or g_z shows transitivity.

Supposing the second condition holds, we can appeal to the fact that $\gamma^2 = \gamma$ to conclude that $\gamma(x) \in [x]_\gamma$, and that therefore there exists some $g \in G$ such that $x = g(\gamma(x))$ regardless of x. This implies the other two items.

Supposing the third condition, we prove the first. Let $x \in V$ be given. By assumption we have only $x = g(\gamma(y))$ for some $g \in G$ and $y \in V$. But, appealing to the G-invariance of γ and using $\gamma^2 = \gamma$,

$$\gamma(x) = \gamma(g(\gamma(y))) = \gamma(\gamma(y)) = \gamma(y).$$

Substitute $\gamma(y) = \gamma(x)$ into $x = g(\gamma(y))$ to obtain $x = g(\gamma(x))$. ☐

So if the other conditions for a normal decomposition system are met and if $\gamma^2 = \gamma$, then for Item 3b it suffices to prove that either $V = G(\gamma(V))$ or that G acts transitively on each equivalence class in V/γ. The proof of the precedent proposition shows that $V = G(\gamma(V))$ is the weaker condition; however, the transitivity condition is what we'll focus on.

In the next section, we will characterize some circumstances in which a Euclidean Jordan algebra becomes a normal decomposition system. The difficulty in such a venture is that to fabricate a normal decomposition system out of only a real inner-product space, one must somehow conjure up both the map γ and a group G that is compatible with γ in the sense of Definition 1. We are fortunate that there is a natural choice for γ in a Euclidean Jordan algebra. The primary obstacle, then, is to define the group G—we seem to have an overabundance of options. But keeping in mind that our goal is to decompose the ambient space into $V = G(\gamma(V))$, we would like G to be as large as possible to minimize the number of γ-equivalence classes. And our G must also satisfy Corollary 1. Combining these two facts leads to a canonical choice of G for any map γ.

Proposition 3. *If V and W are real finite-dimensional inner-product spaces and if $\gamma : V \to W$ is continuous, then*

$$\mathfrak{G} := \left\{ g \in \operatorname{Isom}(V) \;\middle|\; \forall x \in V : g\left([x]_\gamma\right) = [x]_\gamma \right\}$$

is a closed subgroup of $\operatorname{Isom}(V)$.

Proof. That $\mathfrak{G}$ is itself a group follows from Corollary 1, the identity

$$\mathfrak{G} = \operatorname{Isom}(V) \cap \left(\bigcap_{X \in V/\gamma} \operatorname{Aut}(X) \right),$$

and the fact that an intersection of subgroups is a subgroup. To see that $\mathfrak{G}$ is closed, observe first that each $[x]_\gamma \in V/\gamma$ is closed: the map γ is continuous, and $[x]_\gamma = \gamma^{-1}(\{\gamma(x)\})$. Then note that for $g \in \operatorname{Isom}(V)$,

$$g\left([x]_\gamma\right) = [x]_\gamma \iff \left[g\left([x]_\gamma\right) \subseteq [x]_\gamma \;\text{ and }\; g^{-1}\left([x]_\gamma\right) \subseteq [x]_\gamma \right].$$

Since $g \mapsto g^{-1}$ is continuous, this is true of the limit of any sequence in $\mathfrak{G}$. □

Having fixed the map γ, we impose only the necessary conditions on $\mathfrak{G}$ in Proposition 3. As a result, any other group that satisfies the definition of a normal decomposition system must be a subgroup of $\mathfrak{G}$.

3 Euclidean Jordan algebras

Jordan algebras were conceived by the physicist Pascual Jordan in the early 1930s as formalism for quantum mechanics [20]. The alternative at that time was the Copenhagen interpretation, wherein physical observables are represented by real or complex Hermitian matrices. The immediate problem with that interpretation is that those two sets are not closed under the operations of complex scaling and matrix multiplication. Jordan recognized that the "quasi-multiplication" of matrices $x \bullet y := (xy + yx)/2$ is commutative, preserves conjugate-symmetry, and satisfies the identity $x \bullet ((x \bullet x) \bullet y) = (x \bullet x) \bullet (x \bullet y)$. These constitute the axioms of a *Jordan algebra*.

To further specialize, we insist that the algebra be over the reals and that it satisfy the condition—inspired by the Hermitian matrices—that $x \bullet x + y \bullet y = 0$ implies $x = y = 0$. These two requirements imply the existence of a compatible inner product that makes the entire structure a Euclidean Jordan algebra [4].

Definition 2. A *Euclidean Jordan algebra* $(V, \bullet, \langle \cdot, \cdot \rangle)$ consists of a finite-dimensional, real, commutative, unital algebra V whose bilinear "Jordan product" operation $\bullet$ satisfies

$$x \bullet ((x \bullet x) \bullet y) = (x \bullet x) \bullet (x \bullet y) \text{ for all } x, y \in V,$$

and whose inner product satisfies

$$\langle x \bullet y, z \rangle = \langle y, x \bullet z \rangle \text{ for all } x, y, z \in V. \tag{1}$$

The *degree* of an element $x \in V$ is the dimension of the subalgebra it generates, and the *rank* of a Euclidean Jordan algebra is the maximal degree of its elements. The set $K \coloneqq \{x \bullet x \mid x \in V\}$ is the *cone of squares* in V.

Shortly thereafter, Jordan, von Neumann, and Wigner [14] showed that all Euclidean Jordan algebras are—up to isomorphism—a unique orthogonal direct sum of five simple types of algebras. However, none of those simple algebras are nuanced enough to model quantum mechanics, and for that reason Jordan's plan was laid to rest. Interest in finite-dimensional Euclidean Jordan algebras lay dormant for almost half a century aftwerwards. The impetus for their renewed popularity came when Güler [9] noticed that the cones of squares in Euclidean Jordan algebras correspond to the "self-scaled" cones for which Nesterov and Todd [23] devised efficient optimization algorithms. Since then, many classical matrix results have been extended to Euclidean Jordan algebras. Their amenability to attack can perhaps be summarized: Euclidean Jordan algebras are the abstract setting where every element has a convenient spectral decomposition, analogous to and subsuming that of the Hermitian matrices.

Definition 3 (Jordan frame)**.** If $(V, \bullet, \langle \cdot, \cdot \rangle)$ is a Euclidean Jordan algebra, then $c \in V$ is *idempotent* if $c \bullet c = c$. Two idempotents $c_1, c_2 \in V$ are said to be *orthogonal* if $c_1 \bullet c_2 = 0$, since this implies orthogonality with respect to the inner product. A nonzero idempotent c is *primitive* if there do not exist two nonzero idempotents c_1 and c_2 in V such that $c = c_1 + c_2$. The set $\{c_1, c_2, \ldots, c_r\}$ is a *Jordan frame* in V if its elements are pairwise-orthogonal primitive idempotents that sum to the unit element of V.

Theorem 2 (Faraut and Korányi [4], Theorems III.1.1–2)**.** *If $(V, \bullet, \langle \cdot, \cdot \rangle)$ is a Euclidean Jordan algebra of rank r and if $x \in V$, then there exists a Jordan frame $\{c_1, c_2, \ldots, c_r\}$ in V and real numbers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ such that*

$$x = \lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_r c_r.$$

The numbers λ_i are called the eigenvalues of x*, and this decomposition is unique in the following sense: if $\{d_1, d_2, \ldots, d_r\}$ is a Jordan frame in V and if there exist real numbers $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r$ such that*

$$x = \mu_1 d_1 + \mu_2 d_2 + \cdots + \mu_r d_r,$$

then $\mu_i = \lambda_i$ for all i, and

$$\sum_{\{i \mid \lambda_i = t\}} c_i = \sum_{\{i \mid \mu_i = t\}} d_i$$

for any real number t.

Definition 4. If $(V, \bullet, \langle\cdot,\cdot\rangle)$ is a Euclidean Jordan algebra, then the *trace* of an element is the sum of its eigenvalues in the sense of Theorem 2. The bilinear form $(x, y) \mapsto \operatorname{trace}(x \bullet y)$ defines an inner product on V, hereafter referred to as the *canonical trace inner product*, that always satisfies Equation (1).

The trace inner product has the desirable property that the elements of a Jordan frame all have unit norm in the norm it induces. Having defined eigenvalues in a Euclidean Jordan algebra, it is natural to ask if we can define spectral functions as well. That question was first asked and answered affirmatively by Baes [1] who instilled the function $\lambda^{\downarrow}$ with its obvious and more-general meaning on a Euclidean Jordan algebra. The properties of spectral functions on Euclidean Jordan algebras have been studied extensively ever since [26, 25, 13, 7].

Normal decomposition systems capture the essence of the classical spectral functions. Can they do the same for spectral functions on Euclidean Jordan algebras? The following was originally shown by Lim, Kim, and Faybusovich [18] and extended somewhat by Gowda and Jeong [7]. An "essentially-simple" Euclidean Jordan algebra is an algebra that is either simple or $\mathbb{R}^n$. From now on, we let $\operatorname{JAut}(V)$ denote the set of Jordan-algebra automorphisms of V—the subset of $\operatorname{Aut}(V)$ that preserves the Jordan product.

Theorem 3. *If V is an essentially-simple Euclidean Jordan algebra of rank r with the canonical trace inner product, if $G \coloneqq \operatorname{JAut}(V)$, and if*

$$\gamma(x) \coloneqq \lambda_1^{\downarrow}(x)\, c_1 + \lambda_2^{\downarrow}(x)\, c_2 + \cdots + \lambda_r^{\downarrow}(x)\, c_r \tag{2}$$

for some fixed Jordan frame $\{c_1, c_2, \ldots, c_r\}$ in V, then (V, G, γ) forms a normal decomposition system.

The use of the trace inner product in Theorem 3 guarantees that G is a subgroup of $\operatorname{Isom}(V)$. A few remarks on this choice of γ are in order. Recall from Example 1 how $\operatorname{diag}(\lambda^{\downarrow}(X))$ was used to represent the class of matrices whose spectra coincide with that of X. If $\{e_1, e_2, \ldots, e_n\}$ is the standard basis in $\mathbb{R}^n$ and if we define $E_i \coloneqq e_i e_i^T$, then

$$\operatorname{diag}\left(\lambda^{\downarrow}(X)\right) = \lambda_1^{\downarrow}(X)\, E_1 + \lambda_2^{\downarrow}(X)\, E_2 + \cdots + \lambda_n^{\downarrow}(X)\, E_n.$$

This is precisely what the map γ does—literally, in the Euclidean Jordan algebra of real symmetric matrices with the Jordan frame $\{E_1, E_2, \ldots, E_n\}$. So the choice of γ in Theorem 3 is not arbitrary, and is in fact a natural extension of the normal map used with the Hermitian matrices. In particular we note that $[x]_\gamma = [x]_{\lambda^\downarrow}$ for any x, based on Equation (2) and the uniqueness of the eigenvalues in Theorem 2.

Can any other choice of G in Theorem 3 make (V, G, γ) a normal decomposition system when V is *not* essentially-simple? On the contrary, we will show that $G \coloneqq \mathrm{JAut}(V)$ is canonical for the chosen γ.

Lemma 1 (Faraut and Korányi [4], Section III.5)**.** *If V is a Euclidean Jordan algebra with the canonical trace inner product and cone of squares K, then* $\mathrm{JAut}(V) = \mathrm{Aut}(K) \cap \mathrm{Isom}(V)$.

Proposition 4. *If V is a Euclidean Jordan algebra with the canonical trace inner product, if γ is as in Theorem 3, and if (V, G, γ) forms a normal decomposition system, then $G \subseteq \mathrm{JAut}(V)$.*

Proof. Every $g \in G$ satisfies $g\left([x]_\gamma\right) = [x]_\gamma$ for all x by Proposition 1, so g preserves eigenvalues. In particular, if x belongs to the cone of squares K in V, then its eigenvalues are nonnegative by Theorem 2, and the eigenvalues of $g(x)$ are nonnegative too. Therefore $g(K) \subseteq K$, and an easy computation shows that $g^*(K^*) \subseteq K^*$. But K is self-dual [4], and $g^* = g^{-1}$, so we conclude that $g^{-1}(K) \subseteq K$. Thus $g \in \mathrm{Aut}(K)$, and Lemma 1 says that $g \in \mathrm{JAut}(V)$. ☐

Having shown that $G = \mathrm{JAut}(V)$ is canonical for our γ, we augment the following result of Jeong and Gowda with an additional characterization of essentially-simple algebras in terms of $\mathrm{JAut}(V)$.

Theorem 4. *If V is a Euclidean Jordan algebra of rank r, then the following are equivalent:*

1. *V is essentially-simple.*
2. *If $\{c_1, c_2, \ldots, c_r\}$ and $\{d_1, d_2, \ldots, d_r\}$ are any two Jordan frames in V, then there exists some $\phi \in \mathrm{JAut}(V)$ such that $\phi(c_i) = d_i$ for all i.*
3. *The group $\mathrm{JAut}(V)$ acts transitively on each equivalence class in $V/\lambda^\downarrow$.*

Proof. The equivalence of the first two items is Theorem 11 in Jeong and Gowda [13]. The second condition implies the third by linearity after taking spectral decompositions; so suppose the third condition holds, and let $\{c_1, c_2, \ldots, c_r\}$ and $\{d_1, d_2, \ldots, d_r\}$ be two Jordan frames in V. We can define

$$x \coloneqq 1c_1 + 2c_2 + \cdots + rc_r,$$
$$y \coloneqq 1d_1 + 2d_2 + \cdots + rd_r,$$

and by assumption there exists some $\phi \in \mathrm{JAut}(V)$ sending x to y. The uniqueness in Theorem 2 with $t = 1, 2, \ldots, r$ shows that $\phi(c_i) = d_i$ for all i. ☐

We are now ready to prove our main result. Afterwards, it can be appended to the list of equivalent conditions in Theorem 4. We note that Theorem 5.7 in Gowda and Jeong [7] contains yet another equivalent condition involving (weakly) spectral sets.

Theorem 5. *If V is a Euclidean Jordan algebra of rank r with the canonical trace inner product and if γ is defined as in Theorem 3, then there exists a closed subgroup G of* $\operatorname{Isom}(V)$ *making (V, G, γ) a normal decomposition system if and only if V is essentially-simple.*

Proof. When V is essentially-simple, $G := \operatorname{JAut}(V)$ will work. This is merely Theorem 3. Conversely, if V is not essentially-simple, then Proposition 4 shows that any candidate G would have to be a subgroup of $\operatorname{JAut}(V)$. Then by Theorem 4, there would be some $x \in V$ such that G fails to act transitively on $[x]_{\lambda^{\downarrow}} = [x]_{\gamma}$. This would contradict Proposition 2 if (V, G, γ) were a normal decomposition system, so we conclude that it cannot be. □

This technique effectively rules out any function γ where $[x]_{\gamma} = [x]_{\lambda^{\downarrow}}$ for all $x \in V$. Which is not to say that there isn't some other function γ that will do the job, but it is rather damning for the study of spectral functions. Neither does Theorem 5 imply that powerful results cannot be obtained for spectral functions on Euclidean Jordan algebras; it says only that we are unlikely to obtain them by way of a normal decomposition. Gowda was prescient in this regard, having recently invented *Fan-Theobald-von-Neumann systems* [6] to subsume both normal decomposition systems and (non-essentially-simple) Euclidean Jordan algebras. The details of these systems are beyond the present scope; it suffices to say that every normal decomposition system is a Fan-Theobald-von-Neumann system, but that the precise relationship between the two is still unknown.

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