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The Lyapunov rank of an improper cone

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Let K be a closed convex cone with dual K^* in a finite-dimensional real inner-product space V. The complementarity set of K is

$$C(K) = \{(x, s) \in K \times K^* \mid \langle x, s \rangle = 0\}.$$

We say that a linear transformation $L : V \to V$ is Lyapunov-like on K if

$$\langle L(x), s \rangle = 0 \text{ for all } (x, s) \in C(K).$$

The dimension of the space of all such transformations is called the Lyapunov rank of K. This number was introduced and studied by Rudolf et al. [10] for *proper* cones because of its connection to conic programming and complementarity problems. The assumption that K is proper turns out to be nonessential.

We first develop the basic theory for cones that are merely closed and convex. We then devise a way to compute the Lyapunov rank of any closed convex cone and show that the Lyapunov-like transformations on a closed convex cone are related to the Lie algebra of its automorphism group. Next we extend some results for proper polyhedral cones. Finally, we devise algorithms to compute both the space of all Lyapunov-like transformations and the Lyapunov rank of a polyhedral closed convex cone.

Keywords: Lyapunov rank, Lyapunov-like transformation, conic programming, Lie algebra, automorphism group

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1. Introduction

Let K be a closed convex cone in an n-dimensional real inner-product space V with dual

$$K^* = \{y \in V \mid \forall x \in K, \ \langle x, y \rangle \geq 0\}. \tag{1}$$

The complementarity set of K is then

$$C(K) = \{(x, s) \in K \times K^* \mid \langle x, s \rangle = 0\}.$$

Such a set arises in connection with complementarity problems [3] and as optimality conditions in conic programming [2].

It is known that $C(K)$ is an n-dimensional manifold within the $2n$-dimensional space $V \times V$. This inspired Rudolf et al. [10] to investigate the possibility of expressing the single equation $\langle x, s \rangle = 0$ in the complementarity set as a system of n or more independent equations, $\langle L_i(x), s \rangle = 0$ for $i = 1, 2, \ldots, n$. When this can be done, there is hope of solving the system using existing algorithms. To quantify this possibility, the authors introduced the *bilinearity rank* of a cone.

Gowda and Tao [5] then noticed that the bilinearity rank of a cone K can be described in terms of its *Lyapunov-like* transformations, $L : V \to V$ having the property that

$$\langle L(x), s \rangle = 0 \text{ for all } (x, s) \in C(K).$$

Gowda and Tao showed that the bilinearity rank of K is the dimension of the space of all Lyapunov-like transformations on K. These transformations are related to the Lyapunov transformations in the theory of dynamical systems; hence the term 'Lyapunov rank' was coined in place of 'bilinearity rank'. Gowda and Tao also connected the Lyapunov-like transformations on K to the Lie algebra of its automorphism group, showing that L is Lyapunov-like on K if and only if $L \in \mathrm{Lie}(\mathrm{Aut}(K))$.

Example 1 The prototypical Lyapunov-like transformations occur on the cone $K = \mathbb{R}^n_+$, the nonnegative orthant in $\mathbb{R}^n$. The cone $\mathbb{R}^n_+$ is self-dual and generated by the standard basis vectors $\{e_1, e_2, \ldots, e_n\}$. As a result, its complementarity set consists of pairs of standard basis vectors $\{(e_i, e_j) \mid i \neq j\} = C(\mathbb{R}^n_+)$. If L is an $n \times n$ real matrix, then the Lyapunov-like condition $\langle L(e_i), e_j \rangle = L_{ji} = 0$ for all $(e_i, e_j) \in C(\mathbb{R}^n_+)$ implies that every off-diagonal entry of L must be zero. The remaining space of diagonal matrices has a basis $\{e_1 e_1^T, e_2 e_2^T, \ldots, e_n e_n^T\}$ of Lyapunov-like transformations, and thus, the Lyapunov rank of $\mathbb{R}^n_+$ is the size n of that basis. This is reflected in the 'complementary slackness' condition, that $\langle x, s \rangle = 0$ for $x, s \in \mathbb{R}^n_+$ is equivalent to $x_i s_i = 0$ for $i = 1, 2, \ldots, n$.

Recent work has concentrated on bounding and computing the Lyapunov rank for individual cones. Rudolf et al. [10] computed the bilinearity rank of the moment cone. Gowda and Tao [5] derived results for polyhedral and symmetric cones. Gowda, Sznajder, and Tao [4] investigated the completely positive and copositive cones. And Gowda and Trott [6] have considered special Bishop-Phelps cones.

In all previous work, the cones were assumed to be *proper*; that is: closed, convex, pointed (containing no lines), and solid (having nonempty interior). We ask what happens when the cones are merely closed and convex—one can still define the Lyapunov rank as the dimension of the space of all Lyapunov-like transformations on the cone. The assumption that the cones are proper turns out to be nonessential. We develop the theory for closed convex cones and revisit some important results. For the special case of polyhedral cones, we devise algorithms to compute Lyapunov-like transformations and the Lyapunov rank.

The practical motivation for this work is a need to experiment. For example, Gowda and Tao showed that L is Lyapunov-like on a proper cone K if and only if $L \in \mathrm{Lie}(\mathrm{Aut}(K))$. One begins to notice that the same equivalence holds for some important but improper cones. The space of all $n \times n$ real matrices is the Lie algebra of the automorphism group of $\mathbb{R}^n$, and every such matrix is Lyapunov-like on $\mathbb{R}^n$. Similar 'coincidences' happen with subspaces and half-spaces, but the computations quickly become tedious. The need to experiment demands an implementation.

Existing software provides the necessary building blocks for polyhedral closed convex cones. For the cleanest design, any Lyapunov rank algorithm should work in as much generality. And yet, surprises such as the failure of the product formula in Proposition 8 indicate that we cannot be careless. The theory of Section 3 was developed as a solid foundation on which to build an implementation—a naive approach using Algorithm 1—

for improper cones. Experiments then led to the results in Section 4, and the attempt to unify those results became Theorem 2. The techniques of Theorem 2 are ultimately crucial to our second major theorem, the Lie algebra connection for improper cones. Finally, those theoretical techniques pay dividends and improve the naive algorithm in the final section.

2. Preliminaries

2.1 *Standard definitions*

Let V and W be finite-dimensional real inner-product spaces. By $\mathcal{B}(V, W)$ we denote the space of all linear maps from V to W. We abbreviate $\mathcal{B}(V, V)$ by $\mathcal{B}(V)$. The adjoint $L^* \in \mathcal{B}(W, V)$ of $L \in \mathcal{B}(V, W)$ is defined by $\langle L(x), y\rangle_W = \langle x, L^*(y)\rangle_V$ for all $x \in V, y \in W$.

We say that $L \in \mathcal{B}(V)$ is an automorphism of $X \subseteq V$ and write $L \in \operatorname{Aut}(X)$ if L is invertible and $L(X) = X$. If $L \in \mathcal{B}(V, W)$ preserves inner products, we call it an *isomorphism*, and write $K \cong J$ to indicate that $L(K) = J$ or vice-versa. The composition of L_1 with L_2 is written $L_1 \circ L_2$.

For $x, s \in V$, we define $x \otimes s \in \mathcal{B}(V)$ as the map $y \mapsto \langle s, y\rangle x$. From this it follows that $x \otimes s$ has as its adjoint $(x \otimes s)^* = s \otimes x \in \mathcal{B}(V)$. Moreover, $x \otimes L^*(s) = (x \otimes s) \circ L$. Next we define the *trace operator* on $\mathcal{B}(V)$ as the sum-of-eigenvalues, $\operatorname{trace}(L) := \sum_{\lambda\in\sigma(L)} \lambda$. It should be clear that

$$\operatorname{trace}(x \otimes s) = \operatorname{trace}(s \otimes x) = \langle x, s\rangle . \tag{2}$$

On $\mathcal{B}(V)$ we define the trace inner product

$$\langle L_1, L_2\rangle_{\mathcal{B}(V)} := \operatorname{trace}(L_1 \circ L_2^*) . \tag{3}$$

Definition 1 (cone) A *cone* K in V is a nonempty set such that for all $\lambda \geq 0$ in $\mathbb{R}$ we have $\lambda K = K$. A closed convex cone is a cone that is convex and topologically closed.

Our interest is restricted to closed convex cones.

Definition 2 (conic hull) Given a nonempty subset X of V, the *conic hull* of X is

$$\operatorname{cone}(X) := \{\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_m x_m \mid x_i \in X, \alpha_i \geq 0\} .$$

When X is finite, the set $\operatorname{cone}(X)$ is a closed convex cone in V.

Definition 3 (generators) We say that G *generates* K if $\operatorname{cone}(G) = K$. If G generates K, then the elements of G are called *generators* of K.

Definition 4 (span, dimension, lineality) Let K be a closed convex cone. Then $\operatorname{span}(K) = K - K$ is the subspace generated by K, and the dimension $\dim(K)$ is defined to be $\dim(\operatorname{span}(K))$. The *lineality* of K is $\operatorname{lin}(K) := \dim(K \cap -K)$.

2.2 *Cone-space pairs*

In Section 1, some operations on K depend implicitly on the ambient space V. When K is a proper cone, there is no ambiguity—no smaller space contains K. But if K lives in a proper subspace W of V, then we will need to (for example) take the dual of K within

W. The notation (1) does not allow this: the expression 'K^*' is ambiguous when we may think of K as living in more than one ambient space, since 'K^* in V' and 'K^* in W' are two different sets. To avert that ambiguity, we make the following definition.

Definition 5 (cone-space pair) A *cone-space pair* (K, V) is a closed convex cone K paired with a finite-dimensional real inner-product space V containing K.

We avoid the cumbersome pair notation with the following useful device.

Definition 6 Suppose (K, V) is a cone-space pair and W is a subspace of V. Then we can define a new cone-space pair $K_W := (K \cap W, W)$. We extend this 'operation' to cone-space pairs by $(K_W)_U = (K_U)_W = K_{UW} := (K \cap U \cap W, U \cap W)$.

Note that $K_V = (K \cap V, V) = (K, V)$ whenever K is contained in V; this motivates an abuse of notation when we say 'let K_V be a cone-space pair' to mean 'let (K, V) be a cone-space pair'.

The *space* in *cone-space pair* (that is, the subscript, from now on) is mainly a book-keeping tool. Any operation defined on a closed convex cone K in a finite-dimensional real inner-product space V can be defined on the cone-space pair K_V in an obvious way: think of K as living in V, perform the operation, and if necessary, pair the result with the appropriate space. Here are a few important examples.

Definition 7 The dual cone-space pair of K_V is another cone-space pair defined by

$$K_V^* := (\{y \in V \mid \forall x \in K,\ \langle x, y\rangle \geq 0\}, V).$$

We define the codimension of K_V in terms of the orthogonal complement of K in V:

$$\operatorname{codim}(K_V) := \dim(\{y \in V \mid \forall x \in K,\ \langle x, y\rangle = 0\}).$$

We will freely perform operations on cone-space pairs that are traditionally defined only on subsets of vector spaces. There is no ambiguity if the space is treated as an annotation. For example, the function $\phi : V \to W$ acts on a cone-space pair by $\phi(K_V) = \phi(K)_W$. Vector spaces are themselves closed convex cones, but we will not belabour the notation. If W is a subspace of V, we write $W^\perp$ and not $W_V^\perp$ for its orthogonal complement in V.

Definition 8 Two cone-space pairs K_V and J_W are *isomorphic*, written $K_V \cong J_W$, if there exists an inner-product-space isomorphism $\phi : V \to W$ with $\phi(K) = J$. When ϕ is merely invertible and linear we say that the cone-space pairs are *linearly isomorphic*.

Definition 9 The cone-space pair K_V is *pointed* if $K \cap -K = \{0\}$ and *solid* if $\operatorname{span}(K) = V$. A *proper* cone-space pair is both pointed and solid.

The next result appears (in terms of polar cones) as Rockafellar's [9] Corollary 14.6.1.

Proposition 1 *A cone-space pair K_V is pointed if and only if K_V^* is solid. Moreover,* $\operatorname{lin}(K_V) = \operatorname{codim}(K_V^*)$.

Definition 10 The *complementarity set* of the cone-space pair K_V is

$$C(K_V) := \{(x, s) \mid x \in K_V, s \in K_V^*, \langle x, s\rangle = 0\}.$$

The map $L \in \mathcal{B}(V)$ is *Lyapunov-like* on K_V if

$$\langle L(x), s\rangle = 0 \text{ for all } (x, s) \in C(K_V).$$

By $\mathbf{LL}(K_V)$ we denote the space of all Lyapunov-like transformations on K_V. The *Lyapunov rank* of K_V is defined to be $\dim(\mathbf{LL}(K_V))$ and is abbreviated $\beta(K_V)$.

The following fact is a consequence of our definitions.

PROPOSITION 2 *Let K_V be a cone-space pair and suppose that W is a subspace of V containing K. Then $(K_W)^* = (K_V^*)_W$.*

3. Basic theory for closed convex cones

3.1 *Lyapunov-like transformations on generators*

Lemma 4 of Rudolf et al. [10] states that when K_V is a proper cone-space pair, the Lyapunov-like property need only be checked for pairs (x, s) of *extreme vectors* with $x \in \operatorname{Ext}(K_V)$ and $s \in \operatorname{Ext}(K_V^*)$. So when K_V is proper, L is Lyapunov-like on K_V if

$$\langle L(x), s\rangle = 0 \text{ for all } (x, s) \in C(K_V) \cap (\operatorname{Ext}(K_V) \times \operatorname{Ext}(K_V^*)). \tag{4}$$

When K_V is proper, $\operatorname{Ext}(K_V)$ generates K_V by the Krein-Milman theorem. This motivates a similar result for closed convex cones. First we show that, by replacing $\operatorname{Ext}(K_V)$ with generators of K_V, we obtain a formula that works for all closed convex cones. Then we give an example of a cone-space pair for which (4) fails.

PROPOSITION 3 *Let K_V be a cone-space pair. Suppose G_1 generates K_V and G_2 generates K_V^*. Then $L \in \mathbf{LL}(K_V)$ if and only if*

$$\langle L(x), s\rangle = 0 \textit{ for all } (x, s) \in C(K_V) \cap (G_1 \times G_2). \tag{5}$$

Proof. Clearly, if $L \in \mathbf{LL}(K_V)$, then L satisfies (5). So suppose that L satisfies (5) and let $(x, s) \in C(K_V)$ be given. We show that $\langle L(x), s\rangle = 0$. Since G_1 generates K_V and G_2 generates K_V^*, we can write

$$\begin{aligned} x &= \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_\ell x_\ell \\ s &= \gamma_1 s_1 + \gamma_2 s_2 + \cdots + \gamma_m s_m \end{aligned}$$

where each $x_i \in G_1$, $s_j \in G_2$, and $\alpha_i, \gamma_j \geq 0$. Because $(x, s) \in C(K_V)$, we have

$$\langle x, s\rangle = 0 \iff \sum_{i=1}^{\ell} \sum_{j=1}^{m} \langle \alpha_i x_i, \gamma_j s_j\rangle = 0.$$

Notice that $\alpha_i x_i \in K_V$ and $\gamma_j s_j \in K_V^*$, so each term in this sum is zero, and thus,

$$(\alpha_i x_i, \gamma_j s_j) \in C(K_V) \cap (G_1 \times G_2) \text{ for all } i, j.$$

Now by supposition,

$$\langle L(x), s\rangle = \sum_{i=1}^{\ell}\sum_{j=1}^{m} \langle L(\alpha_i x_i), \gamma_j s_j\rangle = 0. \qquad \blacksquare$$

DEFINITION 11 (discrete complementarity set) If G_1 and G_2 generate K_V and K_V^* respectively, we refer to $C(K_V) \cap (G_1 \times G_2)$ as a *discrete complementarity set* of K_V. When K_V is polyhedral, it has a finite discrete complementarity set.

Proposition 3 and a generating set for K_V will often allow us to describe its Lyapunov-like transformations and determine its Lyapunov rank. We illustrate this with an example, showing in the process that (4) no longer suffices in the general case.

EXAMPLE 2 Let K be the xy-plane in $V = \mathbb{R}^3$. Then K_V^* is the z-axis in V, and for K_V and K_V^* we have the respective generating sets

$$G_1 = \left\{(1,0,0)^T, (-1,0,0)^T, (0,1,0)^T, (0,-1,0)^T\right\}, \; G_2 = \left\{(0,0,1)^T, (0,0,-1)^T\right\}.$$

Let $\{E_{ij}\}$ for $i,j = 1,2,3$ be the standard basis in $\mathbb{R}^{3\times 3}$. Using Proposition 3, one can verify that neither E_{31} nor E_{32} is Lyapunov-like on K but that the remaining seven E_{ij} are. Thus, $\beta(K_V) = \dim(\mathbf{LL}(K_V)) = 7$. Note that K_V in this example has no extreme vectors; if we use (4) instead of (5), we conclude incorrectly that every E_{ij} is Lyapunov-like on K_V and that $\beta(K_V) = 9$.

Finding a tight upper bound for the Lyapunov rank of a proper cone is an open problem. In an n-dimensional space, the Lyapunov rank of a proper cone is at most $(n-1)^2$, but that bound may not be tight [8]. The following example shows that, in general, the a priori bound of n^2 can be achieved.

EXAMPLE 3 Let $K = V = \mathbb{R}^n$. Then $K_V^* = \{0\}_V$ and $C(K_V) = K \times \{0\}$, so every $L \in \mathcal{B}(V)$ is Lyapunov-like on K_V and $\dim(\mathcal{B}(V)) = n^2$.

The next two results generalize easily to improper cones. The first is mentioned in passing by Rudolf et al. [10], and the second appears as their Lemma 1.

PROPOSITION 4 *The Lyapunov ranks $\beta(K_V)$ and $\beta(K_V^*)$ are equal.*

Proof. It follows from Definition 10 that $L \in \mathbf{LL}(K_V)$ if and only if $L^* \in \mathbf{LL}(K_V^*)$. The map $L \mapsto L^*$ is an automorphism of $\mathcal{B}(V)$, so $\dim(\mathbf{LL}(K_V)) = \dim(\mathbf{LL}(K_V^*))$. $\blacksquare$

PROPOSITION 5 *Let K_V be a cone-space pair, and let $A : V \to W$ be a linear isomorphism. Then $\beta(K_V) = \beta(A(K_V))$.*

Proof. Since $A(K_V) = A(K)_W$, we first observe that $A(K)_W^* = (A^*)^{-1}(K_V^*)$. Then it is evident that $L \in \mathbf{LL}(K_V) \iff ALA^{-1} \in \mathbf{LL}(A(K_V))$. The result follows from the fact that $L \mapsto ALA^{-1}$ is a linear isomorphism between $\mathcal{B}(V)$ and $\mathcal{B}(W)$. $\blacksquare$

3.2 *The codimension formula*

The codimension formula for a proper cone-space pair $K_{\mathbb{R}^n}$ is

$$\beta\left(K_{\mathbb{R}^n}\right) = \operatorname{codim}\left(\operatorname{span}\left(\left\{ sx^T \;\middle|\; (x,s) \in C\left(K_{\mathbb{R}^n}\right)\right\}\right)\right).$$

It originated with Rudolf et al. [10] as their Proposition 1. An analogous formula holds for all closed convex cones.

THEOREM 1 *Suppose K_V is a cone-space pair, and let G_1 and G_2 be any generating sets of K_V and K_V^* respectively. Then,*

$$\beta\left(K_V\right) = \operatorname{codim}\left(\operatorname{span}\left(\left\{ s \otimes x \mid (x,s) \in C\left(K_V\right) \cap \left(G_1 \times G_2\right)\right\}\right)\right). \tag{6}$$

Proof. From (2) and (3) we obtain

$$\langle L(x), s\rangle = \operatorname{trace}\left(x \otimes L^*(s)\right) = \operatorname{trace}\left(x \otimes s \circ L\right) = \langle x \otimes s, L^*\rangle_{\mathcal{B}(V)}.$$

Thus $\langle L(x), s\rangle = 0$ if and only if $\langle x \otimes s, L^*\rangle_{\mathcal{B}(V)} = 0$, and this is easily seen to be equivalent to $\langle s \otimes x, L\rangle_{\mathcal{B}(V)} = 0$. Now using (5), all of the following are equivalent:

- L is Lyapunov-like on K_V.
- $\langle L(x), s\rangle = 0$ for all $(x,s) \in C\left(K_V\right) \cap \left(G_1 \times G_2\right)$.
- $\langle x \otimes s, L^*\rangle_{\mathcal{B}(V)} = 0$ for all $(x,s) \in C\left(K_V\right) \cap \left(G_1 \times G_2\right)$.
- $\langle s \otimes x, L\rangle_{\mathcal{B}(V)} = 0$ for all $(x,s) \in C\left(K_V\right) \cap \left(G_1 \times G_2\right)$.
- $L \in \operatorname{span}\left(\left\{ s \otimes x \mid (x,s) \in C\left(K_V\right) \cap \left(G_1 \times G_2\right)\right\}\right)^{\perp}$.

Therefore, $\dim\left(\mathbf{LL}\left(K_V\right)\right) = \operatorname{codim}\left(\operatorname{span}\left(\left\{ s \otimes x \mid (x,s) \in C\left(K_V\right) \cap \left(G_1 \times G_2\right)\right\}\right)\right)$. ■

4. Lyapunov ranks of some cone-space pairs

The codimension formula (6) may allow us to compute the Lyapunov rank of a cone-space pair. As an example, we consider the cone given by a vector subspace.

PROPOSITION 6 *Let K_V be a cone-space pair where* $\dim(V) = n$ *and K is an m-dimensional subspace of V. Then $\beta\left(K_V\right) = n^2 - m(n-m)$.*

Proof. Using Proposition 5, we can assume that $V = \mathbb{R}^n$ with the standard basis $\{e_i\}_{i=1}^n$ and that $K = \mathbb{R}^m$. Now $G_1 := \{\pm e_i\}_{i=1}^m$ and $G_2 := \{\pm e_i\}_{i=m+1}^n$ generate K_V and $K_V^* = \mathbb{R}^{n-m}$ respectively. Thus,

$$C\left(K_V\right) \cap \left(G_1 \times G_2\right) = \left\{ \left(\pm e_i, \pm e_j\right) \mid i \leq m;\; m+1 \leq j \leq n \right\}.$$

As $\operatorname{span}\left(\left\{ s \otimes x \mid (x,s) \in C\left(K_V\right) \cap \left(G_1 \times G_2\right)\right\}\right)$ reduces to the span of products of the form $e_j \otimes e_i$, it follows from (6) that $\beta\left(K_V\right) = n^2 - m(n-m)$. ■

Note that this result agrees with Example 2 where $n = 3$, $m = 2$, and $\beta\left(K_V\right) = 7$.

PROPOSITION 7 *Let K_V be a cone-space pair with* $\dim(V) = n$ *and* $K = \operatorname{cone}(\{v\})$ *for some nonzero $v \in V$. Then $\beta\left(K_V\right) = n^2 - n + 1$.*

Proof. Without loss of generality, we can take K to be $\operatorname{cone}(\{e_1\})$ and V to be $\mathbb{R}^n$. Then K_V^* is the right half-space containing e_1 in $\mathbb{R}^n$. It is obvious that $G_1 := \{e_1\}$ generates K_V and $G_2 := \{e_1\} \cup \{\pm e_j \mid j > 1\}$ generates K_V^*. By considering the pairs (e_1, e_2) through (e_1, e_n) in (6), we see that $\beta(K_V) = n^2 - (n-1)$. ∎

Corollary 1 *The Lyapunov rank of any ray, half-space, line, or hyperplane in an n-dimensional real inner-product space is $n^2 - n + 1$.*

Proof. The half-space is dual to a single ray, and we can apply Proposition 7 to the set containing a single ray. The line/hyperplane are also duals, and their complementarity sets differ only in sign from those of the ray/half-space. ∎

Proposition 9 of Rudolf et al. [10] shows that Lyapunov rank is additive on a cartesian product when its factors are proper cone-space pairs.

Proposition 8 *Let K_V and J_W be proper. Then $\beta(K_V \times J_W) = \beta(K_V) + \beta(J_W)$.*

Surprisingly, this does not hold in general. If $K = \operatorname{cone}(\{e_1\})$ in $\mathbb{R}^n$, then informally, K can be written as a product $\operatorname{cone}(\{e_1\})_{\mathbb{R}} \times \{0\}_{\mathbb{R}^{n-1}}$. Apply Proposition 8 to that product:

$$\beta\left(\operatorname{cone}(\{e_1\})_{\mathbb{R}} \times \{0\}_{\mathbb{R}^{n-1}}\right) = \beta\left(\operatorname{cone}(\{e_1\})_{\mathbb{R}}\right) + \beta\left(\{0\}_{\mathbb{R}^{n-1}}\right).$$

Proposition 6 and Proposition 7 give $\beta(\{0\}_{\mathbb{R}^{n-1}}) = (n-1)^2$ and $\beta(\operatorname{cone}(\{e_1\})_{\mathbb{R}}) = 1$, respectively, so $\beta(K_{\mathbb{R}^n}) = 1 + (n-1)^2$. Now apply Proposition 7 directly to $K_{\mathbb{R}^n}$ to obtain $\beta(K_{\mathbb{R}^n}) = n^2 - n + 1$. These two results disagree when $n \geq 2$. The process of writing K as a product can be formalized; therefore, Proposition 8 must be invalid for improper cones.

5. The Lyapunov rank of a closed convex cone

The failure of the product formula in Proposition 8 motivates us to find a similar formula that works for all closed convex cones. In our last example, we informally wrote K_V as a product of two cone-space pairs. The first factor was solid in $\operatorname{span}(K)$, and the second factor was trivial in $\operatorname{span}(K)^\perp$. This is a common theme in what follows.

Proposition 9 *Let K_V be a cone-space pair and let W be a subspace of V containing K. Then V is isomorphic to $W \times W^\perp$, and $K_V \cong K_W \times \{0\}_{W^\perp}$.*

Proof. Suppose $\{e_i\}_{i=1}^m$ and $\{f_j\}_{j=m+1}^n$ are bases for W and $W^\perp$ respectively. Define ϕ by $\phi(e_i) = (e_i, 0)^T$ and $\phi(f_j) = (0, f_j)^T$. Evidently $\phi : V \to W \times W^\perp$ is an inner-product-space isomorphism and $\phi(K_V) = K_W \times \{0\}_{W^\perp}$. ∎

Proposition 9 and Proposition 5 show that we can find $\beta(K_V)$ by computing $\beta(K_W \times \{0\}_{W^\perp})$ instead. When K_V is non-solid, the latter is simpler.

Lemma 1 *Let K_V be a cone-space pair and $S = \operatorname{span}(K)$. Then K_S is solid and*

$$\beta(K_V) = \beta(K_S) + \operatorname{codim}(K_V) \cdot \dim(V).$$

Proof. Through Proposition 9, we can work with $K_S \times \{0\}_{S^\perp}$ instead of K_V. We will connect the Lyapunov-like transformations on K_S to those on $K_S \times \{0\}_{S^\perp}$. First observe that the complementarity sets of these cone-space pairs are related:

$$\forall t \in S^{\perp},\ \left((x,0)^T,(s,t)^T\right) \in C\left(K_S \times \{0\}_{S^{\perp}}\right) \iff (x,s) \in C\left(K_S\right).$$

Now suppose that $L \in \mathbf{LL}\left(K_S \times \{0\}_{S^{\perp}}\right)$ is expressed in block form,

$$\begin{aligned} L &: S \times S^{\perp} \to S \times S^{\perp} \\ L &:= \begin{bmatrix} A & B \\ Z & D \end{bmatrix}, \end{aligned}$$

where $A \in \mathcal{B}\left(S\right)$, $B \in \mathcal{B}\left(S^{\perp}, S\right)$, $Z \in \mathcal{B}\left(S, S^{\perp}\right)$, and $D \in \mathcal{B}\left(S^{\perp}\right)$. Since K_S is solid, we must have $Z = 0$; otherwise we can choose $x \in K$ having $\langle Z\left(x\right), t\rangle \neq 0$ for some $t \in S^{\perp}$ and contradict the Lyapunov-like property of L.

We claim that $L \in \mathbf{LL}\left(K_S \times \{0\}_{S^{\perp}}\right)$ if and only if $A \in \mathbf{LL}\left(K_S\right)$. This is obvious after we note that $\left\langle L\left((x,0)^T\right), (s,t)^T\right\rangle = \langle A\left(x\right), s\rangle$ and we recall the relationship between the two complementarity sets. If we desire an $L \in \mathbf{LL}\left(K_S \times \{0\}_{S^{\perp}}\right)$, then we are free to choose A, B, and D from their respective spaces having dimensions $\beta\left(K_S\right)$, $\dim\left(S^{\perp}\right)\dim\left(S\right)$, and $\dim\left(S^{\perp}\right)^2$. Thus,

$$\beta\left(K_V\right) = \beta\left(K_S\right) + \dim\left(S^{\perp}\right)\left(\dim\left(S\right) + \dim\left(S^{\perp}\right)\right). \qquad \blacksquare$$

Lemma 1 is only half the story—we need to be able to deal with non-pointed cone-space pairs as well. Fortunately these problems are dual to one another.

Lemma 2 *Let K_V be a cone-space pair and $P = \operatorname{span}\left(K_V^*\right)$. Then K_P is pointed and*

$$\beta\left(K_V\right) = \beta\left(K_P\right) + \operatorname{lin}\left(K_V\right) \cdot \dim\left(V\right).$$

Proof. Applying Lemma 1 to K_V^*, we have

$$\beta\left(K_V^*\right) = \beta\left(\left(K_V^*\right)_P\right) + \operatorname{codim}\left(K_V^*\right) \cdot \dim\left(V\right).$$

Now $\left(K_V^*\right)_P$ is solid and, by Proposition 2, equal to K_P^*. If we take its dual and apply Proposition 1, then K_P is pointed. Substituting $\beta\left(K_V^*\right) = \beta\left(K_V\right)$ and $\beta\left(K_P^*\right) = \beta\left(K_P\right)$ by Proposition 4, we obtain the result. $\blacksquare$

The preceding lemmas combine to handle any closed convex cone.

Theorem 2 *Let K_V be a cone-space pair, $S = \operatorname{span}\left(K\right)$, and $P = \operatorname{span}\left(K_S^*\right)$. Then K_{SP} is proper and*

$$\beta\left(K_V\right) = \beta\left(K_{SP}\right) + \operatorname{lin}\left(K\right) \cdot \dim\left(K\right) + \operatorname{codim}\left(K_V\right) \cdot \dim\left(V\right).$$

Proof. Apply Lemma 1 to K_V so that we have

$$\beta\left(K_V\right) = \beta\left(K_S\right) + \operatorname{codim}\left(K_V\right) \cdot \dim\left(V\right) \tag{7}$$

where K_S is solid. Now apply Lemma 2 to K_S:

$$\beta(K_S) = \beta((K_S)_P) + \operatorname{lin}(K_S) \cdot \dim(S) \tag{8}$$

where $(K_S)_P = K_{SP}$ is pointed. The lineality of K_S and dimension of S are the same as those of K itself, so combining (7) and (8), we have

$$\beta(K_V) = \beta(K_{SP}) + \operatorname{lin}(K) \cdot \dim(K) + \operatorname{codim}(K_V) \cdot \dim(V).$$

Since K_S was solid, the cone-space pair K_{SP} is solid (and thus proper) as well. ■

The literature states that for any proper K_V, the identity transformation is Lyapunov-like on K_V and that therefore $\beta(K_V) \geq 1$. However, the trivial cone in the trivial space is both solid and pointed with Lyapunov rank zero. We caution that the K_{SP} obtained in Theorem 2 can be trivial, as our next example shows.

Example 4 Suppose $K = \mathbb{R}^m$ in $V = \mathbb{R}^n$. Then $\operatorname{lin}(K) = \dim(K) = m$, $\operatorname{codim}(K_V) = n - m$, and K_{SP} is trivial. Theorem 2 then gives $\beta(K_V) = n^2 - m(n-m)$.

Example 5 If $K = \operatorname{cone}(\{v\})$ in the n-dimensional space V (cf. Proposition 7), then $\operatorname{lin}(K) = 0$, $\dim(K) = 1$, and $\operatorname{codim}(K_V) = n-1$. If $S = \operatorname{span}(\{v\})$, then the solid cone-space pair K_S is just $\operatorname{cone}(\{v\})_S$ which is self-dual in S. As a result, $P = \operatorname{span}(K_S^*) = S$ and so $K_{SP} = K_S$. Corollary 1 and Theorem 2 give $\beta(K_V) = n^2 - n + 1$.

Example 6 Suppose that K_V is proper. Then $S = P = V$, so $K_{SP} = K_V$ and $\operatorname{lin}(K) = \operatorname{codim}(K_V) = 0$. Theorem 2 reduces to $\beta(K_V) = \beta(K_{SP})$.

Theorem 2 gives us a way to 'shrink' the computation of $\beta(K_V)$ when K_V is improper. Notice that every Lyapunov rank computation reduces to that of a proper cone-space pair K_{SP}. Insofar as Lyapunov rank is concerned, and from a theoretical point of view, this suggests that proper cones are the right objects to study. However, in the next section we will see that Lyapunov-like transformations are interesting even for improper cones.

6. Characterization of Lyapunov-like transformations

The main idea of Section 5 is that the structure of a non-solid cone-space pair K_V lets us describe its Lyapunov-like transformations, and that we can use the dual K_V^* to do the same for non-pointed cone-space pairs. This approach extends to the automorphism group of K_V in order to characterize $\mathbf{LL}(K_V)$. The following interesting connection between $\mathbf{LL}(K_V)$ and the Lie algebra of $\operatorname{Aut}(K_V)$ was made by Gowda and Tao [5].

Theorem 3 *Suppose that K_V is a proper cone-space pair and that $L \in \mathcal{B}(V)$. Then the following are equivalent:*

- *L is Lyapunov-like on K_V.*
- *$e^{tL} \in \operatorname{Aut}(K_V)$ for all $t \in \mathbb{R}$.*
- *$L \in \operatorname{Lie}(\operatorname{Aut}(K_V))$.*

The proof of this fact relies on Theorem 3 of Schneider and Vidyasagar [12] who deal exclusively with proper cones, so we are left wondering whether or not the same result holds more generally. The equivalence $L \in \operatorname{Lie}(G) \iff e^{tL} \in G$ for all $t \in \mathbb{R}$ is a property of matrix groups G whose details are laid out in Section 7.6 of Baker [1]. And to generalize one half of the remaining equivalence is straightforward.

PROPOSITION 10 *Let K_V be a cone-space pair. If $e^{tL} \in \operatorname{Aut}(K_V)$ for all $t \in \mathbb{R}$, then L is Lyapunov-like on K_V.*

Proof. Let $e^{tL} \in \operatorname{Aut}(K_V)$ for all $t \in \mathbb{R}$, and take any $(x, s) \in C(K_V)$. We show that $\langle L(x), s\rangle = 0$; then it follows that L is Lyapunov-like on K_V. First, since $e^{tL}(x) \in K$,

$$\left\langle \left[e^{tL} - I\right](x), s \right\rangle = \left\langle e^{tL}(x), s \right\rangle \geq 0 \text{ for all } t \in \mathbb{R}.$$

Considering only positive values of t, multiplication by $1/t > 0$ has no effect:

$$\left\langle \frac{1}{t}\left[e^{tL} - I\right](x), s \right\rangle \geq 0 \text{ for all } t > 0.$$

Take the limit as $t \to 0$, then,

$$L = \lim_{t \to 0} \left\{ \frac{1}{t}\left[e^{tL} - I\right] \right\} = \left.\frac{d}{dt} e^{tL}\right|_{t=0}$$

giving $\langle L(x), s\rangle \geq 0$. Replace L by $-L$; the same reasoning gives $\langle L(x), s\rangle \leq 0$. ■

For the converse, it remains to be seen that $L \in \mathbf{LL}(K_V)$ implies $e^{tL} \in \operatorname{Aut}(K_V)$.

PROPOSITION 11 *Suppose K_V is a cone-space pair and that $S = \operatorname{span}(K)$. Then the automorphism group of the cone-space pair $K_S \times \{0\}_{S^\perp}$ is*

$$\operatorname{Aut}(K_S \times \{0\}_{S^\perp}) = \left\{ \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \;\middle|\; A \in \operatorname{Aut}(K_S),\ B \in \mathcal{B}\left(S^\perp, S\right),\ D \in \operatorname{Aut}\left(S^\perp\right) \right\}.$$

Proof. Any transformation in the above set is invertible with

$$\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}BD^{-1} \\ 0 & D^{-1} \end{bmatrix}.$$

Inclusion in one direction is now obvious:

$$\begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \begin{bmatrix} K_S \\ \{0\}_{S^\perp} \end{bmatrix} = \begin{bmatrix} A^{-1} & -A^{-1}BD^{-1} \\ 0 & D^{-1} \end{bmatrix} \begin{bmatrix} K_S \\ \{0\}_{S^\perp} \end{bmatrix} = \begin{bmatrix} K_S \\ \{0\}_{S^\perp} \end{bmatrix}.$$

For the other direction, assume that we have an automorphism L in block form,

$$L := \begin{bmatrix} A & B \\ Z & D \end{bmatrix} \in \operatorname{Aut}(K_S \times \{0\}_{S^\perp}).$$

We will show that $Z = 0$, $A \in \operatorname{Aut}(K_S)$, and $D \in \operatorname{Aut}(S^\perp)$. Invertibility of L requires invertibility of A and D, so those facts are immediate.

Suppose that $Z \neq 0$. Then $Z(x) \neq 0$ for some $x \in K_S$, because K_S is solid. Now $(x, 0)^T \in K_S \times \{0\}_{S^\perp}$, but $L\left((x,0)^T\right) \notin K_S \times \{0\}_{S^\perp}$ since it has a nonzero second component. This contradicts the fact that $L \in \operatorname{Aut}(K_S \times \{0\}_{S^\perp})$. Thus, $Z = 0$.

Two cases remain that would preclude L from being an automorphism of $K_S \times \{0\}_{S^\perp}$.

Case 1 $(A(K_S) \nsubseteq K_S)$**:** We have an obvious contradiction in the fact that

$$L(K_S \times \{0\}_{S^\perp}) = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \begin{bmatrix} K_S \\ \{0\}_{S^\perp} \end{bmatrix} = \begin{bmatrix} A(K_S) \\ \{0\}_{S^\perp} \end{bmatrix} \nsubseteq \begin{bmatrix} K_S \\ \{0\}_{S^\perp} \end{bmatrix}.$$

Case 2 $(A^{-1}(K_S) \nsubseteq K_S)$**:** This contradiction is similar but using L^{-1}.

We have contradictions in both cases, so $A \in \operatorname{Aut}(K_S)$. ■

Our inspiration for the converse is the following realization. Recall from Lemma 1 that $\mathbf{LL}(K_S \times \{0\}_{S^\perp})$ is precisely the set,

$$\left\{ \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \;\middle|\; A \in \mathbf{LL}(K_S),\; B \in \mathcal{B}\left(S^\perp, S\right),\; D \in \mathcal{B}\left(S^\perp\right) \right\}. \tag{9}$$

Thus, any $L \in \mathbf{LL}(K_S \times \{0\}_{S^\perp})$ can be exponentiated directly:

$$e^{tL} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}^k = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k & \widetilde{B} \\ 0 & \sum_{k=0}^{\infty} \frac{t^k}{k!} D^k \end{bmatrix} = \begin{bmatrix} e^{tA} & \widetilde{B} \\ 0 & e^{tD} \end{bmatrix}. \tag{10}$$

Here, $\widetilde{B}$ is unknown, but the form of the expression is suggestive.

Lemma 3 *Suppose K_V is a pointed cone-space pair and that $S = \operatorname{span}(K)$. Then the equivalence from Theorem 3 holds for $K_S \times \{0\}_{S^\perp} \cong K_V$.*

Proof. Take any $L \in \mathbf{LL}(K_S \times \{0\}_{S^\perp})$ according to (9). Exponentiate it as in (10), and note that the cone-space pair K_S is proper, so $e^{tA} \in \operatorname{Aut}(K_S)$ by Theorem 3. The transformation e^{tD} is always invertible, and $\widetilde{B}$ is irrelevant by Proposition 11. ■

Now that we have the result for $K_S \times \{0\}_{S^\perp}$, we extend it to any pointed cone-space pair. To dispose of the isomorphism, we use the following facts whose proofs are trivial.

Proposition 12 *Let $K_V \cong J_W$ be isomorphic cone-space pairs with $K_V = \psi(J_W)$. Then* $\operatorname{Aut}(J_W) = \psi \operatorname{Aut}(K_V)\psi^{-1}$, $\mathbf{LL}(J_W) = \psi \mathbf{LL}(K_V)\psi^{-1}$, *and* $e^{\psi L \psi^{-1}} = \psi e^L \psi^{-1}$.

Corollary 2 *Lemma 3 holds for any pointed cone-space pair K_V.*

Proof. Suppose K_V is a pointed cone-space pair. Then we know that we can write $\phi(K_V) = K_S \times \{0\}_{S^\perp}$ where ϕ is an inner-product space isomorphism and K_S is proper. Take any $L \in \mathbf{LL}(K_V)$. Then from Proposition 12, $\phi L \phi^{-1} \in \mathbf{LL}(K_S \times \{0\}_{S^\perp})$. And from Lemma 3,

$$e^{t\phi L \phi^{-1}} = \phi e^{tL} \phi^{-1} \in \operatorname{Aut}(K_S \times \{0\}_{S^\perp}) \text{ for all } t \in \mathbb{R}.$$

Now using Proposition 12 we obtain $e^{tL} \in \operatorname{Aut}(K_V)$ for all $t \in \mathbb{R}$. ■

Corollary 2 takes care of pointed cone-space pairs. For solid pairs, we work with the dual and therefore need a few more identities.

PROPOSITION 13 *Let* K_V *be a cone-space pair. Then* $\operatorname{Aut}(K_V^*) = \{A^* \mid A \in \operatorname{Aut}(K_V)\}$, $\mathbf{LL}(K_V^*) = \{L^* \mid L \in \mathbf{LL}(K_V)\}$, *and* $e^{t(L^*)} = (e^{tL})^*$.

LEMMA 4 *The equivalence from Theorem 3 holds for a solid cone-space pair* K_V.

Proof. The cone-space pair K_V^* is pointed. Applying Corollary 2 to K_V^* we obtain,

$$L^* \in \mathbf{LL}(K_V^*) \iff e^{t(L^*)} \in \operatorname{Aut}(K_V^*).$$

Now apply Proposition 13 to both sides. ■

Using Lemma 4, we obtain a version of Lemma 3 that does not require K_V to be pointed.

LEMMA 5 *Suppose* K_V *is a cone-space pair and that* $S = \operatorname{span}(K)$. *Then the equivalence from Theorem 3 holds for* $K_S \times \{0\}_{S^\perp} \cong K_V$.

Proof. Proceed as in the proof of Lemma 3. After exponentiating, apply Lemma 4 directly to K_S, which is solid. ■

We now finally address the general case.

THEOREM 4 *Let* K_V *be a cone-space pair and* $L \in \mathcal{B}(V)$. *The following are equivalent:*

- *L is Lyapunov-like on K_V.*
- *$e^{tL} \in \operatorname{Aut}(K_V)$ for all $t \in \mathbb{R}$.*
- *$L \in \operatorname{Lie}(\operatorname{Aut}(K_V))$.*

Proof. Write $\phi(K_V) = K_S \times \{0\}_{S^\perp}$ and mimic the proof of Corollary 2, but using Lemma 5 instead of Lemma 3. ■

A similar result appears in Hilgert, Hofmann, and Lawson [7]. The first two items of their Theorem III.1.10 essentially say that $\mathbf{LL}(K_V) = \operatorname{Lie}(\operatorname{Aut}(K_V))$. However, the remaining items suggest that there may be hidden assumptions, and its proof relies on another Theorem I.5.27 which requires the cone to be solid. Nevertheless, their Theorem I.5.17 and Corollary I.5.18 seem to provide the machinery needed to prove the result.

COROLLARY 3 *For any cone-space pair* K_V, *we have* $\beta(K_V) = \dim(\operatorname{Lie}(\operatorname{Aut}(K_V)))$.

7. Polyhedral cone-space pairs

We now restrict ourselves to the class of polyhedral cones.

DEFINITION 12 (polyhedral cone) We say that the cone-space pair K_V is *polyhedral* if there exists a finite set G such that $K = \operatorname{cone}(G)$.

Since polyhedral cone-space pairs are finitely generated, they have a finite discrete complementarity set. The Lyapunov rank of a *proper* polyhedral cone-space pair has been studied by Gowda and Tao [5]. We revisit their results to see what holds in the general case. Later we devise algorithms to compute $\mathbf{LL}(K_V)$ and $\beta(K_V)$ for polyhedral K_V.

7.1 *Miscellaneous results*

First, a negative result. A *reducible* cone-space pair K_V is a cone-space pair that can be written as a Minkowski sum $I_V + J_V$ where

- I and J are nonempty.
- $I \neq \{0\}$ and $J \neq \{0\}$.
- $\operatorname{span}(I) \cap \operatorname{span}(J) = \{0\}$.

An *irreducible* cone-space pair is a cone-space pair that is not reducible.

Proposition 14 (Gowda and Tao, Corollary 5) *Let K_V be a proper polyhedral cone-space pair. Then $\beta(K_V) = 1$ if and only if K_V is irreducible.*

Note that in one dimension, all cone-space pairs are irreducible, since there exist no nontrivial sets whose spans do not overlap. But Proposition 14 cannot hold in general.

Proposition 15 *Let V be a finite-dimensional real inner-product space with* $\dim(V) \geq 2$. *Then there exists an irreducible polyhedral cone-space pair K_V with $\beta(K_V) > 1$.*

Proof. Take any nonzero $v \in V$, and let $K = \operatorname{cone}(\{v\})$. Then K_V is irreducible. To see why, suppose that we can write $K_V = I_V + J_V$ for nonempty sets I and J with $\operatorname{span}(I) \cap \operatorname{span}(J) = \{0\}$. Without loss of generality, $v \in I$, which means that $J = \{0\}$. As a result, K_V is irreducible. Now apply Proposition 7. ■

The next result partially extends Theorem 2 of Gowda and Tao [5].

Theorem 5 *Suppose K_V is a polyhedral cone-space pair with finite generating set G.*

(i) If every element of G is an eigenvector of L, then $L \in \mathbf{LL}(K_V)$.
(ii) If $L \in \mathbf{LL}(K_V)$, then every extreme vector of K_V is an eigenvector of L.

Proof. The first implication follows from the definition of Lyapunov-like and Proposition 3. Gowda and Tao use their Theorem 3 to prove the second implication for proper K_V. To extend that proof, substitute Theorem 4 as needed. ■

Finally, we generalize one aspect of the following theorem.

Theorem 6 (Gowda and Tao, Theorem 3.i.) *For every proper polyhedral cone-space pair K_V in $V = \mathbb{R}^n$, we have $1 \leq \beta(K_V) \leq n$ and $\beta(K_V) \neq n - 1$.*

We noted subsequent to Theorem 2 that the trivial cone-space pair is proper and polyhedral, so we must correct this statement to $0 \leq \beta(K_V) \leq n$ if we allow n to be zero. Example 3 shows that there can be no similar upper bound on the Lyapunov rank of an improper polyhedral cone. A Lyapunov rank of $\dim(V) - 1$ is, however, still forbidden.

Lemma 6 *Let K_V be a polyhedral cone-space pair. Then* $\beta(K_V) \neq \dim(V) - 1$.

Proof. Let $\dim(K) = m$, $\dim(V) = n$, and $\operatorname{lin}(K) = l$. Then, from Theorem 2,

$$\beta(K_V) = \beta(K_{SP}) + n^2 + m(l - n).$$

Now we set $\beta(K_V) = n - 1$, and rule out all three cases for $\beta(K_{SP})$.

Case 1 ($m = n$ and $l = 0$)**:** This gives $\beta(K_{SP}) = n - 1$ which is impossible by Theorem 6 because K_{SP} is polyhedral and proper.

Case 2 ($m = n$ and $l > 0$)**:** Since l is an integer, this gives $\beta(K_{SP}) = n - 1 - ln < -1$.

Case 3 ($m < n$)**:** We can maximize $\beta(K_{SP}) = n - 1 - n^2 + m(n-l)$ over l by setting $l = 0$. Then the largest that $\beta(K_{SP})$ could possibly be is $n - 1 - n^2 < 0$. ■

7.2 *Algorithms*

For polyhedral cone-space pairs, some generating set—and therefore the associated discrete complementarity set—is finite. This allows us to compute both $\mathbf{LL}(K_V)$ and $\beta(K_V)$. Our first algorithm computes $\mathbf{LL}(K_V)$ for any polyhedral cone-space pair K_V. It is based on the proof of the codimension formula (6), from which we recall

$$L \in \mathbf{LL}(K_V) \iff \langle s \otimes x, L\rangle_{\mathcal{B}(V)} = 0 \text{ for all } (x,s) \in C(K_V) \cap (G_1 \times G_2). \tag{11}$$

Let $\operatorname{vec}(A) = x$ and $\operatorname{mat}(x) = A$ be the inverse operations taking a matrix $A \in \mathbb{R}^{n\times n}$ to the vector $x \in \mathbb{R}^{n^2}$ and vice-versa. Then, given matrix representations for $s \otimes x$ and L, the trace inner product $\langle s \otimes x, L\rangle_{\mathcal{B}(V)}$ is equal to $\langle \operatorname{vec}(s\otimes x), \operatorname{vec}(L)\rangle$. We leverage this to compute $\mathbf{LL}(K_V)$ using existing linear algebra routines: finding all L satisfying (11) becomes the computation of an orthogonal complement.

Algorithm 1 Compute a basis for $\mathbf{LL}(K_V)$

Input: A cone-space pair K_V.
Output: A basis for $\mathbf{LL}(K_V)$.

function LL(K_V)
  $G_1 \leftarrow$ a minimal set of generators for K_V
  $G_2 \leftarrow$ a minimal set of generators for K_V^* ▷ obtainable from G_1
  $C \leftarrow \{(x,s) \mid x \in G_1, s \in G_2, \langle x,s\rangle = 0\}$ ▷ discrete complementarity set
  $W \leftarrow \{\operatorname{vec}(s\otimes x) \mid (x,s) \in C\}$
  $B \leftarrow$ a basis for $W^\perp$ ▷ computed via e.g. Gram-Schmidt
  return $\{\operatorname{mat}(b) \mid b \in B\}$
end function

If $\dim(V) = n$, then K will be input as a list of generators—essentially elements of $\mathbb{Q}^n$, tuples of rational numbers. The arithmetic in Algorithm 1 should be exact, so in general it is not possible to normalize the dual generators or basis elements that arise. The resulting basis for $\mathbf{LL}(K_V)$ need not be orthogonal or normalized.

At this point, we have a way to compute $\beta(K_V)$: simply call LL(K_V) and count how many elements we get back. In fact this is the best algorithm known for proper cones. But if K_V is not guaranteed to be proper, Theorem 2 provides a more efficient approach. To use Theorem 2, we need to implement the 'restrict to subspace' map $K_V \mapsto K_W$. Existing routines assume the dimension of V based on the length n of the input generators, and make no provision for operating in a subspace (reminiscent of the problem that necessitated the introduction of cone-space pairs). The difficulty is best illustrated with an example.

Example 7 Suppose $v \in V$ and $(1,1)^T$ is its representation in terms of some basis. Then $K = \operatorname{cone}\left(\left\{(1,1)^T\right\}\right)$ is interpreted as living in $\mathbb{Q}^2$ since its sole generator has two components. Now, $S = \operatorname{span}(K)$ has dimension one, and we would like to compute $\beta(K_S) = 1$ within S. But if we pass $K_{\mathbb{Q}^2}$ to LL(), it operates in $\mathbb{Q}^2$ giving $|\mathrm{LL}(K_{\mathbb{Q}^2})| = 3$ instead. What we need is to represent v as a tuple with one component, and of course this can be done: if we take $\{v\}$ as our basis for the space S, then v has the representation $(1)^T$ with respect to $\{v\}$. In this case it is clear that $\left|\operatorname{LL}\left(\operatorname{cone}\left(\left\{(1)^T\right\}\right)_{\mathbb{Q}^1}\right)\right| = \beta\left(K_S\right) = 1$.

This approach will work insofar as we are interested in the Lyapunov rank. Starting with a basis $\mathbf{s} = \{s_1, s_2, \ldots, s_m\}$ of W, we can (via extension) suppose that $\mathbf{b} = \mathbf{s} \cup \{b_{m+1}, b_{m+2}, \ldots, b_n\}$ is a basis of V. If $v \in V$ is input as an element $v_\mathbf{e} \in \mathbb{Q}^n$ with respect to some basis $\mathbf{e}$, then the change of basis map $\rho : \mathbb{Q}^n \to \mathbb{Q}^n$ defined by $\rho\left(v_\mathbf{e}\right) = v_\mathbf{b}$ is a linear isomorphism. And if $v \in W$, its representation $v_\mathbf{b}$ will only require m components.

Thus, given $K_{\mathbb{Q}^n}$, we are able to perform the operation $K_{\mathbb{Q}^n} \mapsto \rho\left(K\right)_{\mathbb{Q}^m}$. We apply ρ and then drop the zero components, leaving a vector in $\mathbb{Q}^m$. By Proposition 5, $\beta\left(K_W\right) = \beta\left(\rho\left(K\right)_{\mathbb{Q}^m}\right)$ so this does not affect the result.

Algorithm 2 Restrict K_V to W (up to linear isomorphism)

Input: A cone-space pair K_V and a subspace W of dimension m containing K.
Output: A new cone-space pair $J_{\mathbb{Q}^m}$ linearly isomorphic to K_W.

```
function RESTRICT_TO_SPACE(K_V, W)
    B ← a basis for W
    G ← a minimal set of generators for K_V
    J ← ∅
    for x ∈ G do
        w ← the B-coordinates of x          ▷ disregarding coordinates for B^⊥
        J ← J ∪ {w}
    end for
    return cone(J)_{Q^m}
end function
```

We now present an efficient algorithm for calculating the Lyapunov rank of a cone-space pair K_V. At the outset, the dimension n of V is inferred from the length of the generators of K_V. Then $\dim\left(K\right)$ and $\operatorname{lin}\left(K\right)$ are computed using existing linear algebra routines (row reduction and convex polytope intersection, respectively). Theorem 2 is applied, and the Lyapunov rank of K_{SP} is computed using Algorithm 1.

There are three expensive steps in Algorithm 1. The first is the computation of the generators of K_V^*. The standard approach uses the facet normals of K_V, and that problem grows with the number of generators of K_V. The second expensive operation is the combinatoric construction of the discrete complementarity set. Finally, there is the basis computation using a relative of Gram-Schmidt. Since the basis consists of vectorized $n \times n$ matrices, that takes place in $\mathbb{Q}^{n^2}$.

Algorithm 3 is often an improvement over Algorithm 1 because each of those problems is reduced in size. The proper cone-space pair K_{SP} will (if we are lucky) have fewer generators, fewer facets, fewer complementary pairs, and live in a space of smaller dimension than K_V. It is therefore easier to compute the generators of K_{SP}^* than it is for K_V^*. Moreover, the discrete complementarity set of K_{SP} is constructed from two smaller sets than that of K_V. And if K_{SP} lives in a space of dimension $m < n$, then the basis computation takes place over $\mathbb{Q}^{m^2}$ rather than $\mathbb{Q}^{n^2}$. To be fair, we must now compute $\dim\left(K\right)$ and $\operatorname{lin}\left(K\right)$, and we potentially call RESTRICT_TO_SPACE() twice. However, those computations are all fast relative to the potential improvements.

These algorithms have all been implemented in the Sage Mathematics [11] system.

Algorithm 3 Compute the Lyapunov rank of K_V

Input: A cone-space pair K_V.
Output: The Lyapunov rank of K_V.

function BETA(K_V)
  $\beta \leftarrow 0$; $n \leftarrow \dim(V)$; $m \leftarrow \dim(K)$; $l \leftarrow \operatorname{lin}(K)$
  if $m < n$ **then**
    $K_V \leftarrow$ RESTRICT_TO_SPACE$(K_V, \operatorname{span}(K_V))$
    $\beta \leftarrow \beta + (n - m)\, n$ ▷ Lemma 1
  end if
  if $l > 0$ **then**
    $K_V \leftarrow$ RESTRICT_TO_SPACE$(K_V, \operatorname{span}(K_V^*))$
    $\beta \leftarrow \beta + lm$ ▷ Lemma 2
  end if
  return $\beta + |\text{LL}(K_V)|$ ▷ K_V is proper here, so compute $\beta(K_V)$ the hard way
end function

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References

[1] A. Baker, *Matrix Groups*, Springer, London, 2002.
[2] S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, Cambridge, 2009.
[3] R.W. Cottle, J.-S. Pang, and R. Stone, *The Linear Complementarity Problem*, SIAM, Philadelphia, 2009.
[4] M.S. Gowda, R. Sznajder, and J. Tao, *The automorphism group of a completely positive cone and its Lie algebra*, Linear Algebra and its Applications, 438 (2013) 3862–3871.
[5] M.S. Gowda and J. Tao, *On the bilinearity rank of a proper cone and Lyapunov-like transformations*, Mathematical Programming, 147 (2014) 155–170.
[6] M. S. Gowda and D. Trott, *On the irreducibility, Lyapunov rank, and automorphisms of special Bishop-Phelps cones*, Journal of Mathematical Analysis and Applications, 419 (2014) 172–184.
[7] J. Hilgert, K. Hofmann, and J. Lawson. *Lie groups, convex cones and semigroups*, Oxford University Press, Oxford, 1989.
[8] M. Orlitzky and M.S. Gowda, *An improved bound for the Lyapunov rank of a proper cone*, Optimization Letters, 10 (2016) 11–17.
[9] R.T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, 1970.
[10] G. Rudolf, N. Noyan, D. Papp, and F. Alizadeh, *Bilinear optimality constraints for the cone of positive polynomials*, Mathematical Programming, Series B, 129 (2011) 5–31.
[11] The Sage Developers, *Sage Mathematics Software (Version 6.10)*, 2015, http://www.sagemath.org/.
[12] H. Schneider and M. Vidyasagar, *Cross-positive matrices*, SIAM Journal on Numerical Analysis, 7 (1970) 508–519.