Lyapunov rank in conic optimization

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Part 1, Section 1

Intro: Convex optimization

Intro: Convex optimization

Everything takes place in a Hilbert space V:

- V is *finite-dimensional*.
- V is a vector space over the *real numbers*.

It won't hurt to pretend that $V = \mathbb{R}^n$.

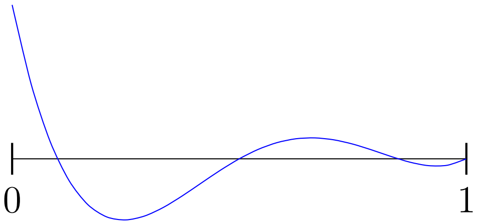
Intro: Convex optimization

Optimization is:

- trying to find the best value of a function,
- or its least-bad value,
- or simply *any* value that works.

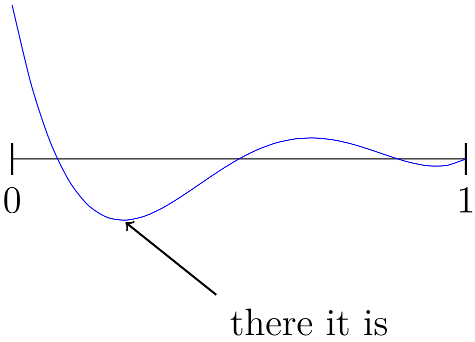
Intro: Convex optimization

Example. Minimize a real function over $[0, 1]$.



Intro: Convex optimization

Example. Minimize a real function over $[0, 1]$.



Intro: Convex optimization

In other words,

$$\begin{array}{ll} \text{minimize} & f(x) = \text{a nice polynomial} \\ \text{subject to} & \quad x \in [0,1]. \end{array}$$

Intro: Convex optimization

Why can we solve it?

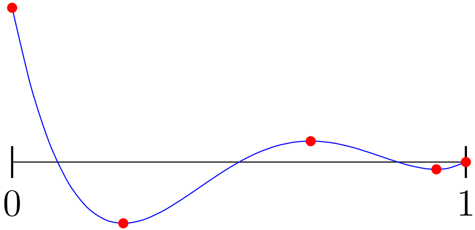
The minimum exists because,

- the function f is continuous, and
- the interval $[0, 1]$ is closed and bounded.

Intro: Convex optimization

We can *find* the minimum because,

- the function f is differentiable,
- the interval $[0, 1]$ is convex, and
- there aren't too many places to look:



Intro: Convex optimization

Definition (convex set).

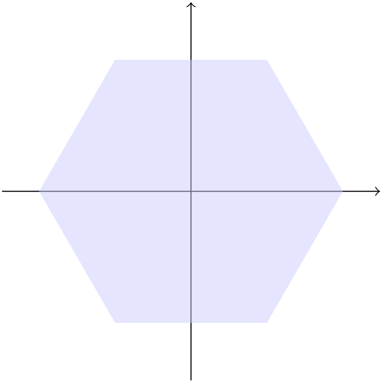
A set is convex if we can

- pick a point x in the set
- pick a point y in the set

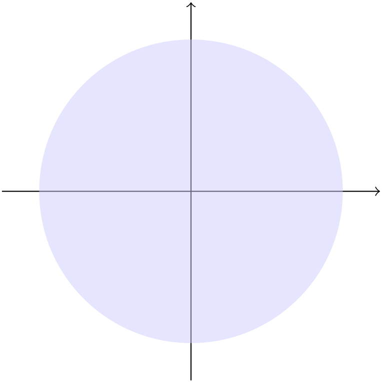
and be sure that

- the line segment joining x and y is in the set

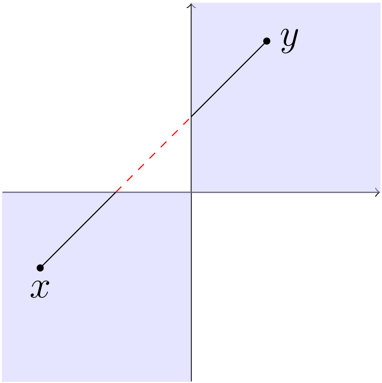
Example (convex).



Example (convex).



Example (not convex).



Intro: Convex optimization

Question. Why convexity?

Answer (via joke).

In optimization we have only two tools,

1. Taylor series
2. Newton's method

Intro: Convex optimization

Often some constraints will make life difficult.

Example.

$$\begin{array}{ll} \text{minimize} & f(x) = \text{a nice polynomial} \\ \text{subject to} & x \in [0,1] \\ \text{and} & x \text{ is rational.} \end{array}$$

Intro: Convex optimization

The constraints can even be the hardest part.

Example.

$$
\begin{aligned}
\text{minimize} \quad & f(x, y, z) = 1 \\
\text{subject to} \quad & x, y, z \in [0, 1] \\
\text{and} \quad & x^3 y^2 z - y^3 = -\sqrt{\pi}, \\
\text{and} \quad & \sin(z) = y \int_0^x \Gamma(t)\ dt, \\
\text{and} \quad & \cdots \text{ make it stop}
\end{aligned}
$$

Intro: Convex optimization

It's real easy to make up impossible problems.

Example.

$$\begin{aligned} &\text{minimize} && f(x) = \text{whatever} \\ &\text{subject to} && x \in [0,1] \\ &\text{and} && x \geq 9000. \end{aligned}$$

Intro: Convex optimization

To keep things manageable, we insist that

- the function f is a *convex function*, and
- we're optimizing over a *convex set*.

That's "convex optimization."

Intro: Cones

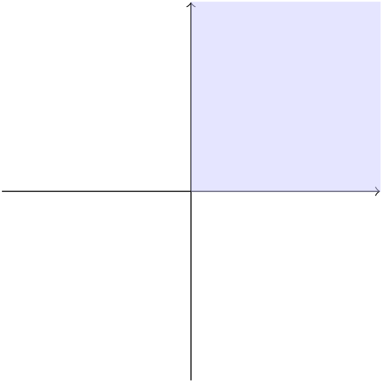
Intro: Cones

Definition.

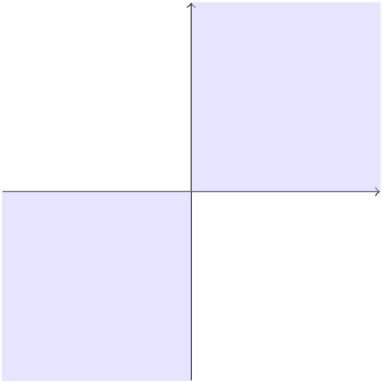
A set K is a cone if $\lambda K \subseteq K$ for all $\lambda \geq 0$.

Intro: Cones

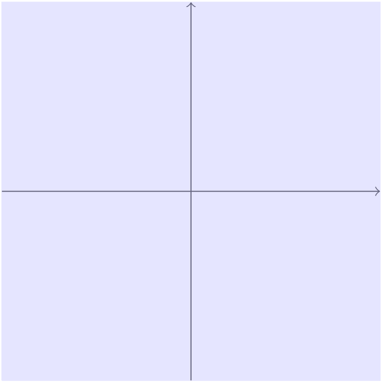
Example (convex cone).



Example (non-convex cone).

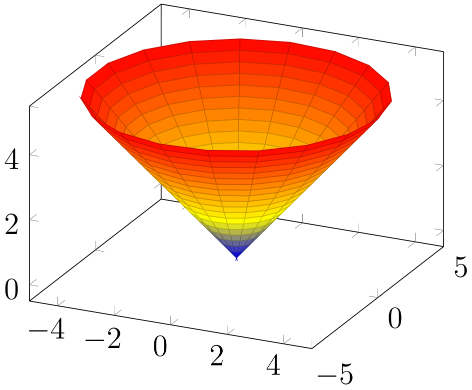


Example (convex cone).



Intro: Cones

Example (nonconvex cone).



Definition (dual cone).

The dual cone of K is

$$K^* := \{y \in V \mid \langle x, y \rangle \geq 0 \text{ for all } x \in K\}.$$

Intro: Cones

Dual cones generalize orthogonal complements:

- the x-axis is a convex cone in $\mathbb{R}^2$
- its dual cone is the y-axis
- but don't worry about it too much

Intro: Cones

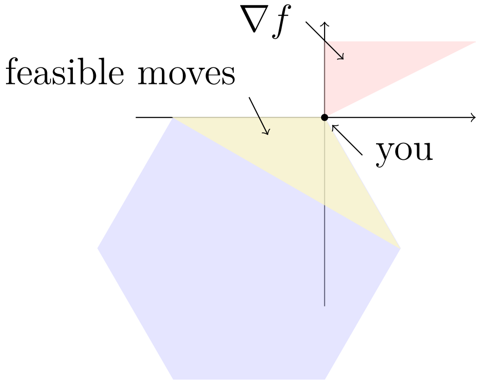
Question. Why (dual) cones?

Answer.

Along the boundary of a convex set, the directions you can go form a (dual) cone.

Intro: Cones

Example (optimality conditions).



Intro: Complementarity

Example (primal linear program).

Given L, b, c; find a vector x to

$$\begin{aligned} \text{minimize} \qquad & \langle b, x \rangle \\ \text{subject to } L\left(x\right) - c & \geq 0 \\ x & \geq 0. \end{aligned}$$

Example (dual linear program).

Simultaneously, find a vector s to

$$\begin{aligned} \text{maximize} \quad & \langle c, s \rangle \\ \text{subject to} \quad & b - L^{T}(s) \geq 0 \\ & s \geq 0. \end{aligned}$$

Intro: Complementarity

If x and s are primal and dual optimal, then $\langle b, x\rangle = \langle c, s\rangle$. Thus by substitution,

$$\underbrace{\langle L(x), s\rangle - \langle c, s\rangle}_{\geq 0} = \underbrace{\left\langle L^T(s), x\right\rangle - \langle b, x\rangle}_{\leq 0}.$$

It follows that

$$\langle s, L(x) - c\rangle = 0 = \left\langle x, L^T(s) - b\right\rangle.$$

Intro: Complementarity

As a result, we always have

$$\langle s, L(x) - c \rangle = 0 = \left\langle x, L^T(s) - b \right\rangle$$

for optimal x and s in the linear program.

This condition is *complementary slackness*.

Example (linear complementarity).

Given $M \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$, the LCP (M, q) is

$$\begin{aligned} \text{to find} \qquad & x, s \geq 0 \\ \text{such that} \qquad & s = M(x) + q \\ \text{and} \qquad & \langle x, s \rangle = 0. \end{aligned}$$

Intro: Complementarity

If we set

$$M := \begin{bmatrix} 0 & -L^T \\ L & 0 \end{bmatrix} \quad \text{and} \quad q := \begin{bmatrix} b \\ -c \end{bmatrix},$$

then LCP (M, q) solves our linear programs.

Intro: Complementarity

A linear complementarity problem can be forumated over a cone K and its dual K^{*}:

$$\begin{array}{ll} \text{find} & x \in K,\ s \in K^{*} \\ \text{such that} & s = M(x) + q \\ \text{and} & \langle x, s \rangle = 0. \end{array}$$

Intro: Complementarity

Why? To solve harder problems:

- robust linear programs
- nonconvex quadratic programs
- graph max-cut

Our goal: solve cone complementarity problems by finding all $x \in K$ and $s \in K^*$ with $\langle x, s \rangle = 0$.

Intro: Complementarity

This technique works even if we replace the linear operator M with a more general f:

$$\begin{aligned} &\text{find} & x &\in K \\ &\text{such that} & f(x) &\in K^{*} \\ &\text{and} & \langle x, f(x) \rangle &= 0. \end{aligned}$$

Intro: Complementarity

Let $C(K)$ be the set of complementary pairs,

$$C(K) \coloneqq \{(x, s) \in K \times K^{*} \mid \langle x, s \rangle = 0\}.$$

Our general complementarity problem is then to

$$\text{find } (x, f(x)) \in C(K).$$

Intro: Lyapunov-like operators

Intro: Lyapunov-like operators

Ok, but how should we find all (x, s) with

- $x \in K$,
- $s \in K^*$, and
- $\langle x, s \rangle = 0$?

Intro: Lyapunov-like operators

We can write id_V in terms of a basis $\{L_i\}$,

$$\mathrm{id}_V = \sum L_i.$$

Then

$$0 = \langle x, s \rangle = \langle \mathrm{id}_V(x), s \rangle = \sum \langle L_i(x), s \rangle.$$

Intro: Lyapunov-like operators

…but so what? The equation

$$\sum \langle L_i(x), s \rangle = 0$$

isn't any easier to solve than $\langle x, s \rangle = 0$.

Idea.

Define some operators that make it easier.

Definition (Lyapunov-like operator).

The linear operator L is *Lyapunov-like* on K if

$$\langle L(x), s \rangle = 0 \text{ for all } (x, s) \in C(K),$$

where you will recall that

$$C(K) := \{(x, s) \in K \times K^{*} \mid \langle x, s \rangle = 0\}.$$

Intro: Lyapunov-like operators

The set of all Lyapunov-like operators on K is denoted by $\mathbf{LL}(K)$.

It turns out that

- $\mathbf{LL}(K)$ is a vector subspace, and
- $\mathbf{LL}(K)$ always contains the identity, id_V.

Intro: Lyapunov-like operators

If $\{L_1, L_2\}$ is a basis of $\mathbf{LL}(K)$, then

$$(x, s) \in C(K)$$

$$\Updownarrow$$

$$(x, s) \in K \times K^* \text{ and } \langle \mathrm{id}_V(x), s \rangle = 0$$

$$\Updownarrow$$

$$(x, s) \in K \times K^* \text{ and } \langle L_i(x), s \rangle = 0 \text{ for } i = 1, 2$$

Intro: Lyapunov-like operators

The definition of Lyapunov-like is exactly what we need to split the single equation

$$\sum_{i=1}^{2} \langle L_i(x), s \rangle = 0$$

into two equations

$$\langle L_1(x), s \rangle = 0$$
$$\langle L_2(x), s \rangle = 0.$$

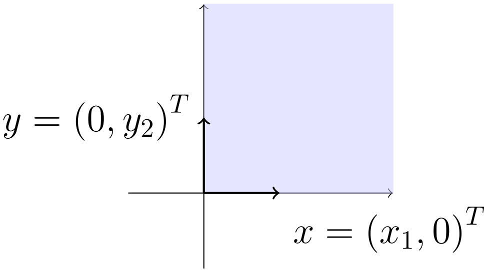
Example.

If K is the quadrant where $x \geq 0$ and $y \geq 0$, then

$$\mathbf{LL}\left(K\right) = \operatorname{span}\left(\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}\right).$$

Example.

These are the only orthogonal $x \geq 0$ and $y \geq 0$:



Example.

It's easy to check that

$$\left\langle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ y_2 \end{bmatrix} \right\rangle = 0$$

and

$$\left\langle \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ y_2 \end{bmatrix} \right\rangle = 0 \ldots$$

Intro: Lyapunov-like operators

Example.

and

$$\left\langle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ y_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \right\rangle = 0$$

and

$$\left\langle \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ y_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \right\rangle = 0.$$

Intro: Lyapunov-like operators

We get $\dim\left(\mathbf{LL}\left(K\right)\right)$ equations from $\langle x, s\rangle = 0$.

The more equations, the better.

We call $\dim\left(\mathbf{LL}\left(K\right)\right)$ the *Lyapunov rank* of K, and denote it by

$$\beta\left(K\right) \coloneqq \dim\left(\mathbf{LL}\left(K\right)\right).$$

Intro: Lyapunov-like operators

Definition (good cone).

K is a "good" cone if $\beta(K) \geq \dim(V)$.

(We get a square system in that case.)

Part 2, Section 5

Results: Lyapunov rank

Results: Lyapunov rank

Theorem (Gowda and Tao, 2014).

All symmetric cones are good cones.

Symmetric means:

- self-dual
- (which implies proper)
- and “homogeneous”

Results: Lyapunov rank

Example.

- $\beta\left(\mathbb{R}^n_+\right) = n$ in $\mathbb{R}^n$
- $\beta\left(\mathcal{L}^n_+\right) = \left(n^2 - n + 2\right)/2$ in $\mathbb{R}^n$
- $\beta\left(\mathcal{S}^n_+\right) = n^2$ in $\mathcal{S}^n$
- $\beta\left(\mathcal{H}^n_+\right) = 2n^2 - 1$ in $\mathcal{H}^n$

Results: Lyapunov rank

Corollary.

There exist non-self-dual good cones.

Proof.

$\beta(K_1 \times K_2) = \beta(K_1) + \beta(K_2)$ for proper K_1, K_2.
Pick $K_1 = \mathcal{H}^n_+$ and K_2 asymmetric. $\square$

Results: Lyapunov rank

Theorem (Sznajder, 2016).

There exist *irreducible* non-self-dual good cones.

(That is, not using the cartesian product trick.)

Results: Lyapunov rank

Question.

How does homogeneity affect Lyapunov rank?

Guess.

Makes it bigger.

Results: Lyapunov rank

Theorem (Gowda and Tao, 2014).

If K is proper and polyhedral, then

$$\beta\left(K\right) \le \dim\left(V\right).$$

Results: Lyapunov rank

Theorem (Gowda and Tao, 2014).

If K is proper and polyhedral, then

$$L \text{ is Lyapunov-like on } K$$

$$\iff$$

$$L(x) = \lambda x \text{ for all } x \in \operatorname{Ext}(K).$$

Results: Lyapunov rank

Theorem (Gowda and Tao, 2014).

If K is proper and polyhedral, then

$$\beta(K) = 1 \iff K \text{ is irreducible.}$$

Reducible means "into a nontrivial direct sum."

Results: Lyapunov rank

Proposition (Orlitzky, 2017).

If K is a closed convex cone, then

$$\beta\left(K^{*}\right) = \beta\left(K\right).$$

Results: Lyapunov rank

Proposition (Orlitzky, 2017).

If K is a closed convex cone, then

$$\beta\left(L\left(K\right)\right) = \beta\left(K\right)$$

for any invertible linear operator L.

Results: Lyapunov rank

Theorem (Orlitzky, 201X).

If K is a polyhedral closed convex cone, then $\mathbf{LL}(K)$ is closed under composition.

Results: Lyapunov rank

Lemma (Orlitzky, 2017).

If $K = \operatorname{cone}(G_1)$ and if $K^* = \operatorname{cone}(G_2)$, then

$$L \text{ is Lyapunov-like on } K$$
$$\iff$$
$$\langle L(x), s\rangle = 0 \text{ for all } x \in G_1$$
$$\text{and } s \in G_2 \text{ with } \langle x, s\rangle = 0.$$

We can check a polyhedral cone in finite time.

Results: Lyapunov rank

Theorem (Orlitzky, 2017).

If K is a closed convex cone in V, then

$$\beta(K) = \beta(K_{SP}) + \operatorname{lin}(K)\dim(K) + \operatorname{codim}(K)\dim(V).$$

where K_{SP} is a proper subcone of K in an appropriate subspace.

Results: Lyapunov rank

The previous theorem provides a shortcut for computing the Lyapunov rank of improper cones.

```
sage: K = random_cone(); K
12-d cone in 34-d lattice N
sage: timeit('K.lyapunov_like_basis()')
5 loops, best of 3: 10.8 s per loop
sage: timeit('K.lyapunov_rank()')
5 loops, best of 3: 289 ms per loop
```

Results: Lyapunov rank

Theorem (Orlitzky, 2017).

If K is a polyhedral closed convex cone in V, then

$$\beta(K) \neq \dim(V) - 1.$$

Results: Lyapunov rank

Theorem (Orlitzky, 2017).

If K is a closed convex cone and if L is linear, then the following are equivalent:

- L is Lyapunov-like on K.
- $e^{tL} \in \operatorname{Aut}(K)$ for all $t \in \mathbb{R}$.
- $L \in \operatorname{Lie}(\operatorname{Aut}(K))$.

Results: Lyapunov rank

Definition (copositive operator).

The linear operator L is *copositive* on K if

$$\langle L\left(x\right), x\rangle \geq 0 \text{ for all } x \in K$$

The set of copositive operators on K is $\mathbf{CoP}\left(K\right)$.

Example. The PSD matrices are $\mathbf{CoP}\left(\mathbb{R}^n\right)$.

Results: Lyapunov rank

Theorem (Gowda, Sznajder, Tao, 2013).

If K is a proper cone, then

$$\beta\left(\mathbf{CoP}\left(K\right)\right) = \beta\left(K\right).$$

RESULTS: LYAPUNOV RANK

Definition (positive operator).

The linear operator L is *positive* on K if

$$L(K) \subseteq K.$$

The set of all positive operators on K is $\pi(K)$.

Example. Nonnegative matrices are $\pi(\mathbb{R}^n_+)$.

Results: Lyapunov rank

Theorem (Orlitzky, 201X).

If K is a proper polyhedral cone, then

$$\beta\left(\pi\left(K\right)\right) = \beta\left(K\right)^{2}.$$

Question.

What about nonpolyhedral K?

Results: Lyapunov rank

Theorem (Orlitzky and Gowda, 2016).

If K is a proper cone in V, then

$$\beta(K) \leq (\dim(V) - 1)^2.$$

This is “easy” with a lemma.

Results: Lyapunov rank

Lemma (Orlitzky and Gowda, 2016).

If

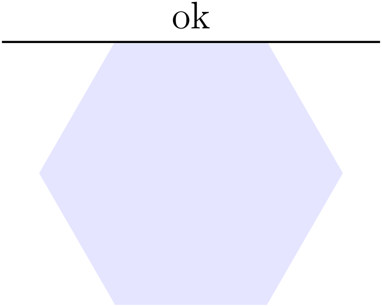
- K is proper
- $H_1, H_2, \ldots, H_N$ are hyperplanes,
- $\operatorname{bdy}(K)$ is a subset of $\cup H_i$,

then K is polyhedral.

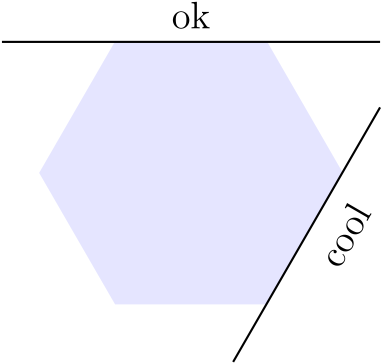
Results: Lyapunov rank

Take a cross-section of a proper cone:

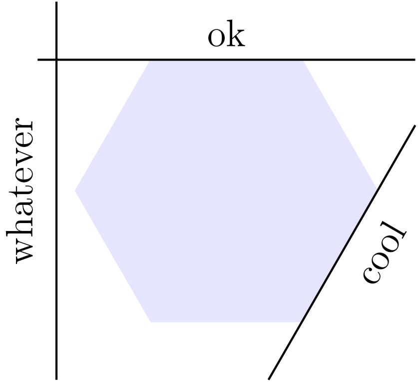
Results: Lyapunov rank



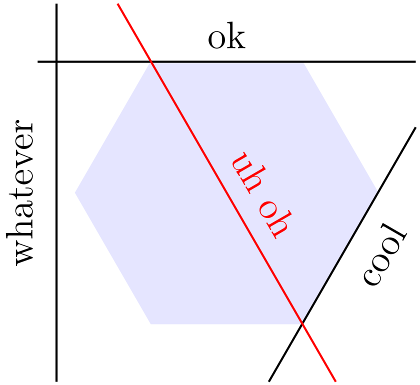
Results: Lyapunov rank



Results: Lyapunov rank



Results: Lyapunov rank



Results: Lyapunov rank

If there aren't any red planes:

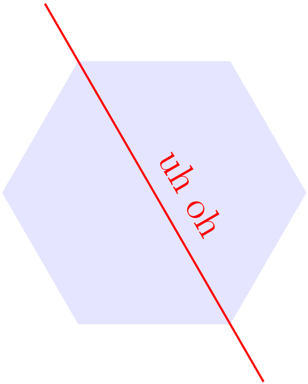
- the planes generate a cone
- that cone is polyhedral by definition
- and it equals K (our hexagon)

Results: Lyapunov rank

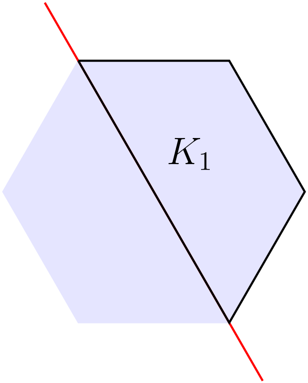
One or more red planes:

- kill one
- now there's one fewer
- use recursion

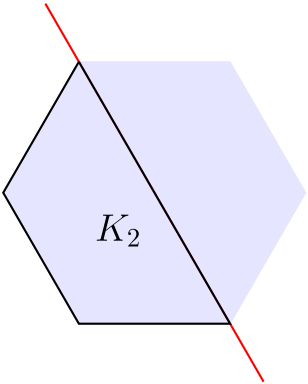
Results: Lyapunov rank



Results: Lyapunov rank



Results: Lyapunov rank



Results: Lyapunov rank

Now using convexity,

$$K = K_1 \cup K_2,$$

and the red plane doesn't hurt K_1 or K_2.

So,

- the result holds for K_1 and K_2,
- K is polyhedral if both K_1 and K_2 are.