Ornstein-Uhlenbeck Processes

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**Introduction**

**Goal.** To introduce a new financial derivative.

- No fun.
- I’m bad at following directions.
- The derivatives based on Geometric Brownian Motion don’t model reality anyway.

So we’re going to replace the underlying stochastic process instead.
Claim. GBM doesn’t work in real life.

Proof.
**Wiener Processes**

**Definition (Wiener Process).** You should know this already [1] (Chapter 4).

Recall two important properties:

1. \( W_0 = 0 \).
2. \( s < t \) implies \( [W_t - W_s] \sim N [0, \sqrt{t - s}] \)

**Note.** As the time period \( t - s \) approaches infinity, so does the standard deviation!
Wiener Processes

\[ W_t := W_{t_0} + \int_0^t \sigma(s) \, dW_s \]

\( W_t \) is a Wiener process with \( \sigma \) being a deterministic function.
What’s wrong here?

• If you’re unlucky, the Wiener process can go bonkers.

• When it does, it tends to stay there.

This doesn’t accurately reflect what happens in real life.
The Black-Scholes model that we’ve been using assumes that the stock dynamics follow Geometric Brownian Motion:

\[ dX_t = \alpha X_t dt + \sigma X_t dW_t \]

\[ X_0 = x_0 \]

with solution,

\[ X_t = x_0 \exp \left\{ \left( \alpha - \frac{\sigma^2}{2} \right) t + \sigma W_t \right\} \]

Since this contains \( W_t \), it inherits all of its problems.
Geometric Brownian Motion

\[ \alpha = 1 \]
\[ \sigma = 1/5 \]
\[ x_0 \exp \left\{ (\alpha - \frac{\sigma^2}{2})t + \sigma W_t \right\} \]
\[ x_0 \exp \left\{ (\alpha - \frac{\sigma^2}{2})t \right\} \]
Ornstein-Uhlenbeck

Definition (Ornstein-Uhlenbeck Process). The Ornstein-Uhlenbeck process is a stochastic process with dynamics,

\[ dU_t = \theta (\mu_t - U_t) \, dt + \sigma dW_t \]
\[ U_0 = u_0 \]

where \( W_t \) is a Wiener process.

- Can be seen as a modification of a Wiener process.
- \( \mu_t \) is the mean of the process.
- \( \theta \) is the tendency of the process to return to the mean.
\[ \theta = 2 \]
\[ \sigma = 1/5 \]
\[ \mu = 1 \]
\[ dU_t = \theta (\mu - U_t) dt + \sigma dW_t \]
\[ dU_t = \theta (\mu - U_t) dt + \sigma dW_t \]

- \( \theta = 2 \)
- \( \sigma = 1/5 \)
- \( \mu = 1/2 \)
\[ \theta = 10 \]
\[ \sigma = \frac{1}{5} \]
\[ \mu = \frac{1}{2} \]
\[ dU_t = \theta (\mu - U_t) dt + \sigma dW_t \]
**Theorem.** When $\mu$ is a constant, the solution to the Ornstein-Uhlenbeck SDE is given by,

$$U_t = \mu \left[ 1 - e^{-\theta t} \right] + x_0 e^{-\theta t} + \sigma N [0, \xi]$$

where,

$$\xi = \sqrt{\frac{1}{2\theta} \left[ 1 - e^{-2\theta t} \right]}$$
Proof.

- Look up the solution on Wikipedia.
- Assume that it’s correct.
- Then, the solution is of the form,

\[ f(x_t, t) = x_t e^{\theta t} \]

- Apply Itô’s formula.

(or you can apply Proposition 5.3 from our textbook with 
\( A = -\theta \) and \( b_t = \mu \theta \))
Ornstein-Uhlenbeck

$\theta = 2$
$\sigma = 1/5$
$\mu = 1$

$U_t$
That’s great, but we want to price some derivatives.

Let $P_t$ be some price process, and $p_t = \ln(P_t)$. We’re going to assume that $p_t$ has the dynamics,

$$dp_t = (-\theta (p_t - \eta t) + \eta) dt + \sigma dW_t$$

$$p_0 = p_0$$

We see that $p_t$ is shifted by $\eta t$ – we want to get rid of that.
Lo & Wang Process

We can rewrite that SDE as,

\[ d (p_t - \eta t) = -\theta (p_t - \eta t) \, dt + \sigma dW_t \]

Now if we let \( \eta t = \mu_t \), the right-hand side looks like the Ornstein-Uhlenbeck SDE!

\[ d (p_t - \mu_t) = -\theta (p_t - \mu_t) \, dt + \sigma dW_t \]
Lo & Wang Process

We’ll rename $p_t - \mu_t = q_t$ to reduce the amount of ugly. Thus we have,

$$dq_t = -\theta q_t dt + \sigma dW_t$$

Compare with Geometric Brownian Motion:

$$d \ln (X_t) = \left( \alpha - \frac{\sigma^2}{2} \right) dt + \sigma dW_t$$

Since we have a function of $t$ in the $dt$ term, our process is actually a slight generalization of GBM. It allows us to model trends and predictability.
Lo & Wang Process

Oh, and we can still solve our new process explicitly [3]:

\[ q_t = e^{-\theta t} q_0 + \sigma \int_0^t e^{-\theta (t-s)} dW_s \]

Translating back into the original price process...

\[ p_t = \mu_t + e^{-\theta t} (p_0 - \mu_0) + \sigma N [0, \xi] \]

where again,

\[ \xi = \sqrt{\frac{1}{2\theta} [1 - e^{-2\theta t}]} \]
\begin{align*}
\mu_t &= 0 \\
\theta &= 1 \\
\sigma &= 1/5 \\
\text{If } &\text{ } P_0 \exp \{ \mu_t + e^{-\theta t} (p_0 - \mu_0) + \sigma N[0, \xi] \} \\
\text{then } &\text{ } P_0 \exp \{ \mu_t + e^{-\theta t} (p_0 - \mu_0) \} 
\end{align*}
Lo & Wang Process

\[
\begin{align*}
\mu_t &= t \\
\theta &= 1 \\
\sigma &= 1/5 \\
-p_0 \exp\left\{ \mu_t + e^{-\theta t} (p_0 - \mu_0) + \sigma N[0, \xi] \right\} \\
-p_0 \exp\left\{ \mu_t + e^{-\theta t} (p_0 - \mu_0) \right\}
\end{align*}
\]
Lo & Wang Process

\[\mu_t = 0 \]
\[\theta = 5 \]
\[\sigma = 1/5 \]
\[P_0 \exp\{\mu_t + e^{-\theta t} (p_0 - \mu_0) + \sigma N[0, \xi]\}\]
\[- P_0 \exp\{\mu_t + e^{-\theta t} (p_0 - \mu_0)\}\]
\[ \mu_t = 3t \]
\[ \theta = 5 \]
\[ \sigma = 1 \]

\[ P_0 \exp \{ \mu_t + e^{-\theta t} (p_0 - \mu_0) + \sigma N[0, \xi] \} \]

\[ P_0 \exp \{ \mu_t + e^{-\theta t} (p_0 - \mu_0) \} \]
\[
\begin{align*}
\mu_t &= 5t \\
\theta &= 10 \\
\sigma &= 1/5 \\
\exp\{\mu_t + e^{-\theta t} (p_0 - \mu_0) + \sigma N[0, \xi] \} - \exp\{\mu_t + e^{-\theta t} (p_0 - \mu_0) \}
\end{align*}
\]
Theorem. The Black-Scholes formula works for the process we defined earlier, \( q_t \).

This sounds rather amazing at first (see [3] for a citation), but there is a caveat. While the formula still works, the numerical value of \( \sigma \) will differ between GBM and Ornstein-Uhlenbeck.

The short version:

\[
\sigma^2_{OU} = \sigma^2_{GBM} \cdot \left[ \theta \tau \left( 1 - e^{-\theta \tau} \right)^{-1} \right]
\]

(Here, \( \tau \) is the length of the time period over which your observations are made.)
Option Pricing

Example (IBM, March 2012).

*Table*: Observed prices for March 2012

<table>
<thead>
<tr>
<th>Date</th>
<th>Closing Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>2012-03-01</td>
<td>whatever</td>
</tr>
<tr>
<td>2012-03-02</td>
<td>whatever</td>
</tr>
<tr>
<td>2012-03-05</td>
<td>whatever</td>
</tr>
<tr>
<td>2012-03-06</td>
<td>whatever</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>2012-03-30</td>
<td>whatever</td>
</tr>
</tbody>
</table>
\[ \theta = 1 \]

- \( F_{GBM}(t = 0, s) \)
- \( F_{OU}(t = 0, s) \)
Option Pricing

\[ F_{OU} - F_{GBM} \]

\[ t = 0 \]
\[ s = 205 \]

\[ \theta \]

\[ t = 0 \]
\[ s = 205 \]
\[ F_{OU} - F_{GBM} \]
As $\theta$ increases, the tendency to vary from the mean decreases. Therefore, a large $\theta$ corresponds to a “predictable” stock.

Conclusion. Options on a predictable stock are more valuable!


