Positive operators, Z-operators, Lyapunov rank, and linear games on closed convex cones

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Part 1, Section 1

Introduction: Definitions

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Everything takes place in a Hilbert space V:

- V is *finite-dimensional*.
- V is a vector space over the *real numbers*.

Introduction: Definitions

Definition (cones).

A *cone* is a set K such that $\lambda K \subseteq K$ for all $\lambda \geq 0$.

A closed convex cone is a cone that is closed and convex as a set.

A *proper* cone is a closed convex cone that has interior and contains no lines.

Definition (dual cone).

The dual cone K^* of a set K is

$$K^* := \{y \in V \mid \langle y, x \rangle \geq 0 \text{ for all } x \in K\},$$

The dual cone is always a closed convex cone.

If K is a proper cone, then K^* is too.

Introduction: Definitions

Definition (positive operator).

The linear operator L is *positive* on K if

$$L(K) \subseteq K.$$

The set of all positive operators on K is $\pi(K)$.

Example. Nonnegative matrices are $\pi(\mathbb{R}^n_+)$.

Introduction: Definitions

Definition (complementarity set).

The *complementarity set* of a cone K is,

$$C\left(K\right) \coloneqq \left\{(x, s) \in K \times K^{*} \mid \langle x, s \rangle = 0\right\}.$$

Introduction: Definitions

Definition (Z-operator).

The linear operator L is a ***Z**-operator* on K if

$$\langle L(x), s \rangle \leq 0 \text{ for all } (x, s) \in C(K).$$

The set of all **Z**-operators on K is $\mathbf{Z}(K)$.

Introduction: Definitions

Definition (Lyapunov-like operator).

The linear operator L is *Lyapunov-like* on K if

$$\langle L(x), s \rangle = 0 \text{ for all } (x, s) \in C(K),$$

The set of all Lyapunov-like operators on K is

$$\mathbf{LL}(K) = -\mathbf{Z}(K) \cap \mathbf{Z}(K).$$

Introduction: Definitions

Definition (Lyapunov rank).

The *Lyapunov rank* of K is

$$\beta(K) = \dim(\mathbf{LL}(K)).$$

Introduction: Definitions

Motivation.

A cone complementarity problem is to

$$\text{find } \left(x, f\left(x\right)\right) \in C\left(K\right).$$

This condition gives us $\beta\left(K\right)$ equations.

Part 2, Section 2

Results: Positive & Z-operators

Results: Positive & Z-operators

Theorem (Tam, 1977).

If K is a proper cone, then

$$\pi(K)^* = \operatorname{cone}(\operatorname{Ext}(K^*) \otimes \operatorname{Ext}(K)).$$

Theorem (Orlitzky, 201X).

If $K = \operatorname{cone}(G_1)$ and if $K^* = \operatorname{cone}(G_2)$, then

$$\pi(K)^* = \operatorname{cone}(\{s \otimes x \mid (x, s) \in G_1 \times G_2\}).$$

Results: Positive & Z-operators

The algorithm is available in SageMath:

```
sage: K = Cone([ (1,0), (0,1) ])
sage: K.positive_operators_gens()
[
[1 0]  [0 1]  [0 0]  [0 0]
[0 0], [0 0], [1 0], [0 1]
]
```

Results: Positive & Z-operators

Theorem (Schneider & Vidyasagar, 1970).

If K is a proper cone, then $\pi(K)$ is too.

Theorem (Orlitzky, 201X).

If K is a closed convex cone, then K is proper if and only if $\pi(K)$ is proper.

Results: Positive & Z-operators

Theorem (Tam, 1977).

If K is a proper cone, then K is polyhedral if and only if $\pi(K)$ is polyhedral.

Theorem (Orlitzky, 201X).

If K is a closed convex cone, then K is polyhedral if and only if $\pi(K)$ is polyhedral.

Results: Positive & Z-operators

Theorem (Orlitzky, 201X).

If $K = \operatorname{cone}(G_1)$ and if $K^* = \operatorname{cone}(G_2)$, then $\mathbf{Z}(K)^*$ is the *conic* hull of

$$\left\{-s \otimes x \mid (x, s) \in G_1 \times G_2 \text{ and } \langle x, s \rangle = 0\right\}.$$

This suggests an algorithm to find $\mathbf{Z}(K)$.

Results: Positive & Z-operators

The algorithm is available in SageMath:

```
sage: K = Cone([ (1,0), (0,1) ])
sage: K.Z_operators_gens()
[
[ 0 -1]  [ 0  0]  [-1  0]
[ 0  0], [-1  0], [ 0  0],

[ 1  0]  [ 0  0]  [ 0  0]
[ 0  0], [ 0 -1], [ 0  1]
]
```

Results: Positive & Z-operators

Theorem (Orlitzky, 201X).

If K is a closed convex cone, then K is polyhedral if and only if $\mathbf{Z}(K)$ is polyhedral.

Corollary.

If K is a closed convex cone, then $\pi(K)$ is polyhedral if and only if $\mathbf{Z}(K)$ is polyhedral.

Results: Positive & Z-operators

Theorem (Orlitzky, 201X).

If K is a closed convex cone, then

$$\dim\left(\pi\left(K\right)\right) = \dim\left(\mathbf{Z}\left(K\right)\right).$$

"Obvious" for proper K, but not in general.

Results: Positive & Z-operators

Theorem (Schneider & Vidyasagar, 1970).

If K is proper in $\mathbb{R}^n$ and if $A \in \mathbb{R}^{n \times n}$, then

$$A \in \mathbf{Z}(K) \iff e^{-tA} \in \pi(K) \text{ for all } t \geq 0.$$

Theorem (Orlitzky, 201X).

If K is a closed convex cone and L is linear, then

$$L \in \mathbf{Z}(K) \iff e^{-tL} \in \pi(K) \text{ for all } t \geq 0.$$

Part 2, Section 3

Results: Improper cone rank

Results: Improper cone rank

Lemma (Rudolf et al., 2011).

If K is a proper cone, then

$$L \text{ is Lyapunov-like on } K$$
$$\iff$$
$$\langle L(x), s \rangle = 0 \text{ for all } x \in \operatorname{Ext}(K)$$
$$\text{and } s \in \operatorname{Ext}(K^{*}) \text{ with } \langle x, s \rangle = 0.$$

RESULTS: IMPROPER CONE RANK

Lemma (Orlitzky, 2017).

If $K = \operatorname{cone}(G_1)$ and if $K^* = \operatorname{cone}(G_2)$, then

$$L \text{ is Lyapunov-like on } K$$
$$\iff$$
$$\langle L(x), s \rangle = 0 \text{ for all } x \in G_1$$
$$\text{and } s \in G_2 \text{ with } \langle x, s \rangle = 0.$$

We can check a polyhedral cone in finite time.

Results: Improper cone rank

Most basic results go through to the general case, but the following (surprisingly) does not.

Proposition (Rudolf et al., 2011).

If K and H are proper cones, then

$$\beta\left(K \times H\right) = \beta\left(K\right) + \beta\left(H\right).$$

Results: Improper cone rank

Theorem (Orlitzky, 2017).

If K is a closed convex cone in V, then

$$\begin{aligned}\beta(K) = \beta(K_{SP}) &+ \operatorname{lin}(K)\dim(K)\\ &+ \operatorname{codim}(K)\dim(V).\end{aligned}$$

where K_{SP} is a proper subcone of K in an appropriate subspace.

Results: Improper cone rank

The previous theorem provides a shortcut for computing the Lyapunov rank of improper cones.

Input: A cone K
Output: The Lyapunov rank of K

$$\beta \leftarrow 0$$
$$n \leftarrow \dim(V)$$
$$m \leftarrow \dim(K)$$
$$l \leftarrow \operatorname{lin}(K)$$

Results: Improper cone rank

if $m < n$ **then**

$K \leftarrow \textsc{restrict}\left(K, \operatorname{span}\left(K\right)\right)$

$\beta \leftarrow \beta + \left(n - m\right) n$

end if

if $l > 0$ **then**

$K \leftarrow \textsc{restrict}\left(K, \operatorname{span}\left(K^{*}\right)\right)$

$\beta \leftarrow \beta + lm$

end if

return $\beta + \operatorname{card}\left(\textsc{ll}\left(K\right)\right)$ $\quad\triangleright$ K is proper here

Results: Improper cone rank

And when K is polyhedral, we can run it.

```
sage: K = random_cone(); K
12-d cone in 34-d lattice N
sage: timeit('K.lyapunov_like_basis()')
5 loops, best of 3: 10.8 s per loop
sage: timeit('K.lyapunov_rank()')
5 loops, best of 3: 289 ms per loop
```

Results: Improper cone rank

Theorem (Gowda and Tao, 2014).

If K is a proper polyhedral cone in $\mathbb{R}^n$, then

$$1 \leq \beta(K) \leq n \text{ and } \beta(K) \neq n - 1.$$

Theorem (Orlitzky, 2017).

If K is a polyhedral cone in V, then

$$\beta(K) \neq \dim(V) - 1.$$

Results: Improper cone rank

Theorem (Gowda and Tao, 2014).

If K is a proper cone and if L is linear, then the following are equivalent:

- L is Lyapunov-like on K.
- $e^{tL} \in \operatorname{Aut}(K)$ for all $t \in \mathbb{R}$.
- $L \in \operatorname{Lie}(\operatorname{Aut}(K))$.

Results: Improper cone rank

Theorem (Orlitzky, 2017).

If K is a closed convex cone and if L is linear, then the following are equivalent:

- L is Lyapunov-like on K.
- $e^{tL} \in \operatorname{Aut}(K)$ for all $t \in \mathbb{R}$.
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Part 2, Section 4

Results: Lyapunov rank bound

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Theorem (Gowda and Tao, 2014).

If K is a proper cone in an n-dimensional space, then $\beta(K) \leq n^2 - n$.

Results: Lyapunov rank bound

Theorem (Orlitzky and Gowda, 2016).

If K is a proper cone in an n-dimensional space, then $\beta(K) \leq (n-1)^2$.

The proof involves constructing an additional $n-1$ elements of $\mathbf{LL}(K)^{\perp}$, beyond the n that Gowda and Tao constructed.

Results: Lyapunov rank bound

The theorem relies on a Lemma:

Lemma (Orlitzky and Gowda, 2016).

If K is a proper cone and if its boundary is contained in a *finite* union of hyperplanes, then K is polyhedral.

Part 2, Section 5

Results: Game theory

Results: Game theory

A two-person zero-sum matrix game involves

- Two players,
- Two strategies x, y in the unit simplex Δ,
- A matrix to determine the payoff.

The payoff function with respect to $A \in \mathbb{R}^{n \times n}$ is,

$$(x, y) \mapsto \langle Ax, y \rangle \,.$$

Results: Game theory

The first player wants to maximize $\langle Ax, y \rangle$, and the second player wants to minimize it.

The set Δ is compact and $\langle A\cdot, \cdot \rangle$ is bilinear, so

$$\min_{y \in \Delta} \max_{x \in \Delta} \langle Ax, y \rangle$$

exists, as von Neumann showed.

Results: Game theory

Matrix games...

1. Can be solved by linear programs (von Neumann, 1944).
2. And conversely (Dantzig in the 1960s).
3. Conversely, really (Adler in 2013).

Results: Game theory

Thus game theory can provide insight into optimization problems.

So motivated, Gowda and Ravindran generalized matrix games to linear games.

Results: Game theory

A linear game has two players choosing x and y from a compact base of a self-dual cone K,

$$x, y \in \Delta := \{z \in K \mid \langle z, e \rangle = 1\}.$$

Here, $e \in \operatorname{int}(K)$ ensures that Δ is compact.

The payoff is with respect to a linear operator L:

$$(x, y) \mapsto \langle L(x), y \rangle.$$

Results: Game theory

Gowda and Ravindran (2015) connect linear games to cone complementarity problems.

In particular,

- If L is Lyapunov-like on K, or
- If L is **Z**-operator on K,

then the game's value has nice properties.

Results: Game theory

Orlitzky (201X) extends things to $K \neq K^*$.

The single strategy set Δ is replaced by Δ_1 and Δ_2—one set for each player—defined in terms of $e \in \operatorname{int}(K)$ and $e^* \in \operatorname{int}(K^*)$.

Most results of Gowda and Ravindran generalize.

Results: Game theory

Theorem (Orlitzky, 201X).

If,

- the value of a linear game is zero, and
- $\bar{y} \in \operatorname{int}(K^*)$ for every optimal pair $(\bar{x}, \bar{y})$,

then the optimal pair is unique and $\bar{x} \in \operatorname{int}(K)$.

Results: Game theory

Why generalize?

By one of our theorems, player one wants to

$$\begin{aligned} \text{maximize} \quad & \nu \\ \text{subject to} \quad & x \in K \\ & \langle x, e^* \rangle = 1 \\ & \nu \in \mathbb{R} \\ & L(x) - \nu e \in K \end{aligned}$$

Results: Game theory

Definition.

The primal cone program in standard form is,

$$\begin{array}{ll} \text{minimize} & \langle b, z \rangle \\ \text{subject to} & M(z) - c \in K_2 \\ & z \in K_1 \end{array}$$

where K_1 and K_2 are closed convex cones.

Results: Game theory

Theorem (Orlitzky, 201X).

Player one is trying to solve a cone program, and player two is trying to solve its dual.

(Proof by clever substitution)

Results: Game theory

We can't solve cone programs in general.

But we *can* solve some *symmetric* cone programs.

Corollary.

If K is a symmetric cone, then the associated game is solved by a symmetric cone program.

Results: Game theory

This brings us back to the setting of Gowda and Ravindran, albeit with two strategy sets Δ_1 and Δ_2 instead of just Δ.

But it lets us solve linear games numerically.

Results: Game theory

For example, rock-paper-scissors. . .

```
>>> L = [ [ 0, 1,-1],
...       [-1, 0, 1],
...       [ 1,-1, 0] ]
>>> K = NonnegativeOrthant(3)
>>> e2 = e1 = [1,1,1]
>>> G = SymmetricLinearGame(L,K,e1,e2)
```

Results: Game theory

```
>>> print(G.solution())
Game value: 0.0000000
Player 1 optimal:
  [0.3333333]
  [0.3333333]
  [0.3333333]
Player 2 optimal:
  [0.3333333]
  [0.3333333]
  [0.3333333]
```