The Lyapunov rank of an improper cone

Part I - Algorithms

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Introduction

The study of Lyapunov rank was initiated by Rudolf et al. when they introduced the concept of the bilinearity rank [1] of a cone.

The Lyapunov rank measures how many independent equations $\langle L_i(x), s \rangle = 0$ one can obtain from the single equation $\langle x, s \rangle = 0$ when x, s belong to dual cones.

Introduction

The greater the Lyapunov rank, the more likely it is that we can split the equation $\langle x, s \rangle = 0$ into a solvable system.

For example, the nonnegative orthant in $\mathbb{R}^n$ has Lyapunov rank n. When both x and s are nonnegative, we can split $\langle x, s \rangle = 0$ into n equations $x_1 s_1 = 0$, $x_2 s_2 = 0$, etc.

Introduction

For computing the Lyapunov rank, Rudolf et al. provide the following.

1. The Lyapunov-like property need only be checked for extreme vectors x, s. This can reduce the problem to a finite computation.

2. In $\mathbb{R}^n$, the orthogonal complement of the space of all Lyapunov-like transformations can be constructed from the matrices sx^T.

Introduction

The setting for all of this work is a finite-dimensional real inner-product space V containing a cone K.

Existing work focuses on *proper* cones: closed convex cones that are both pointed and solid. Our goal is to extend some results for proper cones and to compute the Lyapunov ranks of closed convex cones.

Notation

Definition (conic hull).

Given a nonempty subset X of V, the *conic hull* of X is

$$\operatorname{cone}(X) \coloneqq \left\{ \sum \alpha_i x_i \;\middle|\; x_i \in X, \alpha_i \geq 0 \right\}.$$

When X is finite, the set $\operatorname{cone}(X)$ is a closed convex cone in V.

Notation

Definition (generators).

We say that a set G *generates* the cone K if $\operatorname{cone}(G) = K$. If G generates K, then the elements of G are called *generators* of K.

Example. The set $G = \left\{ (1,0)^T, (-1,0)^T \right\}$ generates the x-axis in $\mathbb{R}^2$.

Notation

Caveat: unlike extreme vectors, generators can be redundant. For example, the set

$$G = \left\{ (1,0)^T, (26,0)^T, (-1,0)^T \right\}$$

also generates the x-axis in $\mathbb{R}^2$.

Notation

A generating set specifies a closed convex cone.

```
sage: K = Cone( [(1,0), (-1,0), (0,1)] )
sage: K
2-d cone in 2-d lattice N
sage: K.rays()
N( 0, 1),
N( 1, 0),
N(-1, 0)
in 2-d lattice N
```

Notation

Definition (dimension, lineality).

Let K be a closed convex cone.

The *dimension of* K is $\dim(\operatorname{span}(K))$.

The *lineality of* K is the dimension of the largest subspace contained within K. It is written $\operatorname{lin}(K) \coloneqq \dim(K \cap -K)$.

NOTATION

Definition (cone-space pair).

A *cone-space pair* (K, V) is a closed convex cone K paired with a finite-dimensional real inner-product space V containing K.

If W is an inner-product space, we write K_W to mean [2] the cone-space pair $(K \cap W, W)$.

Notation

Why? Suppose K is contained in a subspace W of V, and we want to take the dual of K within W. How do we write that?

We need the subscript operation anyway, so we repurpose it as a bookkeeping tool. The subscript acts as an annotation to remind the reader where the cone lives.

Notation

We can now formally define familiar concepts.

Definition. The dual cone-space pair of K_V is

$$K_V^* := \left(\{ y \in V \mid \forall x \in K,\ \langle x, y \rangle \geq 0 \}, V \right).$$

Definition. The function $\phi : V \to W$ acts by

$$\phi \left(K_V \right) = \phi \left(K \right)_W .$$

Notation

Definition (pointed,solid,proper). The cone-space pair K_V is *pointed* if $\operatorname{lin}(K_V) = 0$ and *solid* if $\operatorname{span}(K) = V$. A *proper* cone-space pair is both pointed and solid.

```
sage: K = Cone([(1,0)])
sage: K.lineality()
0
sage: K.is_proper()
False
```

Notation

Any operation on a closed convex cone K can be extended to K_V in an obvious way:

1. Think of K as living in V.
2. Perform the operation.
3. If necessary, pair the result with the appropriate space.

Notation

For example, the next result is well-known.

Proposition. The cone-space pair K_V is pointed if and only if K_V^* is solid. Moreover, $\operatorname{lin}(K_V) = \operatorname{codim}(K_V^*)$.

The operations on K_V are nothing but the usual ones on K in the space V.

Notation

Definition (complementarity set).

The *complementarity set* of K_V is

$$C(K_V) := \left\{(x, s) \mid x \in K_V, s \in K_V^*, x \perp s\right\}.$$

There's nothing new except that we use $x \in K_V$ to mean that x is in the "cone part" of K_V.

Notation

Definition (lyapunov-like).

By $\mathcal{B}(V)$ we denote the space of all linear transformations on V. The map $L \in \mathcal{B}(V)$ is *Lyapunov-like* on K_V if

$$\langle L(x), s \rangle = 0 \text{ for all } (x, s) \in C(K_V).$$

Notation

Definition (lyapunov rank).

By $\mathbf{LL}(K_V)$ we denote the space of all Lyapunov-like transformations on K_V. The *Lyapunov rank* of K_V is defined to be the dimension of this space and is abbreviated

$$\beta(K_V) \coloneqq \dim(\mathbf{LL}(K_V)).$$

Basic theory

Proposition. L is Lyapunov-like on K_V if and only if it satisfies the Lyapunov-like property on two generating sets G_1 of K_V and G_2 of K_V^*.

Proof.

Clearly, if $L \in \mathbf{LL}\left(K_V\right)$, then L satisfies the Lyapunov-like property on the generating sets (since the generators belong to the cone).

Proof (continued).

Suppose that G_1 generates K_V, that G_2 generates K_V^*, and that L has the Lyapunov-like property on orthogonal pairs in $G_1 \times G_2$. For any $(x, s) \in C\left(K_V\right)$ we can write

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_\ell x_\ell$$
$$s = \gamma_1 s_1 + \gamma_2 s_2 + \cdots + \gamma_m s_m$$

where each $x_i \in G_1$, $s_j \in G_2$, and $\alpha_i, \gamma_j \geq 0$.

Proof (continued).

Because $(x, s) \in C(K_V)$, we have

$$\langle x, s \rangle = 0 \iff \sum_{i=1}^{\ell} \sum_{j=1}^{m} \langle \alpha_i x_i, \gamma_j s_j \rangle = 0.$$

Notice that $\alpha_i x_i \in K_V$ and $\gamma_j s_j \in K_V^*$, so each term in this sum is zero.

Proof (continued).

But $\langle \alpha_i x_i, \gamma_j s_j \rangle = 0$ means that $(\alpha_i x_i, \gamma_j s_j)$ are pairs of orthogonal generators, and we assumed that L is Lyapunov-like on those pairs. By linearity,

$$\langle L(x), s \rangle = \sum_{i=1}^{\ell} \sum_{j=1}^{m} \langle L(\alpha_i x_i), \gamma_j s_j \rangle = 0.$$

$\square$

Basic theory

This proposition will sometimes allow us to compute the Lyapunov rank.

Example. Let K be the xy-plane in $V = \mathbb{R}^3$. Then K_V^* is the z-axis in V, and they have the respective generating sets

$$\begin{aligned} G_1 &= \left\{ (\pm 1, 0, 0)^T, (0, \pm 1, 0)^T \right\} \\ G_2 &= \left\{ (0, 0, \pm 1)^T \right\}. \end{aligned}$$

Example (continued).

Let $E_{ij} = (\delta_{ij})$ for $i, j = 1, 2, 3$ be the standard basis elements in $\mathbb{R}^{3\times 3}$.

By testing pairs of generators, one can verify that neither E_{31} nor E_{32} is Lyapunov-like on K but that the remaining seven E_{ij} are. Thus, $\beta\left(K_V\right) = 7$.

Basic theory

Example. Let $K = V = \mathbb{R}^n$.

Then $K_V^* = \{0\}_V$ and $C(K_V) = K \times \{0\}$, so every $L \in \mathcal{B}(V)$ is Lyapunov-like on K_V:

$$\langle L(x), 0 \rangle = 0 \text{ for all } (x, 0) \in C(K_V).$$

Therefore, $\beta(K_V) = \dim(\mathcal{B}(V)) = n^2$.

Two more results for proper cones carry over. Their proofs do not become any more interesting when considering cone-space pairs.

Proposition. $\beta\left(K_V\right) = \beta\left(K_V^*\right)$.

Proposition. $\beta\left(K_V\right) = \beta\left(A\left(K_V\right)\right)$ for any invertible $A \in \mathcal{B}\left(V, W\right)$.

Theorem (codimension formula).

Let G_1 and G_2 generate K_V and K_V^*. Then the Lyapunov rank of K_V is

$$\operatorname{codim}\left(\operatorname{span}\left(\left\{s \otimes x \mid (x, s) \in C\left(K_V\right)\right\}\right)\right),$$

and in fact we need only consider pairs of generators, $(x, s) \in C\left(K_V\right) \cap \left(G_1 \times G_2\right)$.

Proof.

Think of $s \otimes x$ as sx^T in $\mathbb{R}^{n \times n}$; the following are all equivalent by properties of the trace:

- $\langle L(x), s \rangle = 0$.
- $\langle x \otimes s, L^* \rangle_{\mathcal{B}(V)} = 0$.
- $\langle s \otimes x, L \rangle_{\mathcal{B}(V)} = 0$.
- $L \in \operatorname{span}(\{s \otimes x\})^{\perp}$.

$\square$

Basic theory

From this equivalence, we can compute $\mathbf{LL}\left(K_V\right)$. Let $\operatorname{vec}\left(A\right) = x$ and $\operatorname{mat}\left(x\right) = A$ be the inverse operations taking a matrix $A \in \mathbb{R}^{n \times n}$ to the vector $x \in \mathbb{R}^{n^2}$ and vice-versa.

If we are given matrix representations of L and $s \otimes x$, we can write them both as long vectors. The computation of $\mathbf{LL}\left(K_V\right)$ then reduces to finding an orthogonal complement.

Basic theory

Input: A cone-space pair K_V.
Output: A basis for $\mathbf{LL}(K_V)$.

$$
\begin{aligned}
&G_1 \leftarrow \text{a generating set for } K_V\\
&G_2 \leftarrow \text{a generating set for } K_V^*\\
&C \leftarrow \{(x, s) \mid x \in G_1, s \in G_2, \langle x, s \rangle = 0\}\\
&W \leftarrow \{\operatorname{vec}(s \otimes x) \mid (x, s) \in C\}\\
&B \leftarrow \text{ a basis for } W^\perp\\
&\textbf{return } \{\operatorname{mat}(b) \mid b \in B\}
\end{aligned}
$$

Basic theory

When K_V is polyhedral, we can actually run this.

```
sage: K=Cone([(1,1,0),(1,-1,1),(-1,0,0)])
sage: K.LL()
[
[1 0 0]  [ 0  1  0]  [ 0  0  1]
[0 1 0]  [ 0  1  2]  [ 0  0 -1]
[0 0 1], [ 0  0 -1], [ 0  0  1]
]
```

Basic theory

This gives the Lyapunov rank, albeit slowly. The product formula [1] suggests an improvement.

Proposition. Let K_V and J_W be proper cone-space pairs. Then

$$\beta\left(K_V \times J_W\right) = \beta\left(K_V\right) + \beta\left(J_W\right).$$

Basic theory

But the product formula doesn't hold in general!

```
sage: K = Cone([(1,0)])
sage: len( K.LL() )
3
sage: len( K.cartesian_product(K).LL() )
10
```

This motivates the search for another formula.

Reduction formula

The trick is to write a non-solid cone as the product of a solid cone and a trivial cone.

Proposition. Let K_V be a cone-space pair and let W be a subspace of V containing K. Then $V \cong W \times W^{\perp}$ and $K_V \cong K_W \times \{0\}_{W^{\perp}}$.

Reduction formula

Lemma 1. Let K_V be a cone-space pair and $S = \operatorname{span}(K)$. Then K_S is solid and

$$\beta(K_V) = \beta(K_S) + \operatorname{codim}(K_V) \cdot \dim(V).$$

Proof. Through isomorphism,

$$\beta(K_V) = \beta\left(K_W \times \{0\}_{W^\perp}\right).$$

Proof (continued).

Elements of $K_W \times \{0\}_{W^\perp}$ look like $(x, 0)$, so its complementarity set is easy to describe:

$$\left((x,0)^T, (s,t)^T\right) \in C\left(K_S \times \{0\}_{S^\perp}\right)$$

$$\Updownarrow$$

$$(x,s) \in C\left(K_S\right).$$

Reduction formula

Proof (continued).

Knowing the complementarity set makes it easy to describe Lyapunov-like transformations:

$$\mathbf{LL}\left(K_S \times \{0\}_{S^\perp}\right) = \left\{ \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \right\}$$

$$A \in \mathbf{LL}\left(K_S\right)$$

$$B = \text{whatever}$$

$$D = \text{whatever}$$

Reduction formula

Proof (continued).

Adding up the dimensions of their respective spaces, we have

$$\beta\left(K_V\right) = \overbrace{\beta\left(K_S\right)}^{A} + \overbrace{\dim\left(S^{\perp}\right)\dim\left(S\right)}^{B} + \underbrace{\dim^2\left(S^{\perp}\right)}_{D}.$$

$\square$

Reduction formula

What about cones that aren't pointed? Just apply the lemma to the dual!

Lemma 2. Let K_V be a cone-space pair and $P = \operatorname{span}\left(K_V^*\right)$. Then K_P is pointed and

$$\beta\left(K_V\right) = \beta\left(K_P\right) + \operatorname{lin}\left(K_V\right) \cdot \dim\left(V\right).$$

Reduction formula

Proof.

Apply Lemma 1 to K_V^* then substitute $\beta(K_V)$ for $\beta(K_V^*)$ and $\beta(K_P)$ for $\beta(K_P^*)$. $\square$

Reduction formula

If we combine lemmas, we reduce the Lyapunov rank computation to that of a proper cone.

Theorem 3. Let K_V be a cone-space pair, $S = \operatorname{span}(K)$, and $P = \operatorname{span}(K_S^*)$. Then K_{SP} is proper and

$$\beta(K_V) = \beta(K_{SP}) + \operatorname{lin}(K) \cdot \dim(K) + \operatorname{codim}(K_V) \cdot \dim(V).$$

Reduction formula

Proof.

Apply Lemma 1 to K_V, and then apply Lemma 2 to the resulting K_S. Note that the lineality of K_S and dimension of S are the same as those of K.

Since K_S was solid, the cone-space pair K_{SP} is solid too. Thus it is proper. $\square$

Reduction formula

Beware: K_{SP} may be trivial with $\beta(K_{SP}) = 0$.

Example 4. Suppose $K = \mathbb{R}^m$ in $V = \mathbb{R}^n$.

Then K_{SP} is trivial, $\operatorname{lin}(K) = \dim(K) = m$, and $\operatorname{codim}(K_V) = n - m$. Theorem 3 gives

$$\beta(K_V) = n^2 - m(n-m).$$

Example 5. Suppose $K = \mathbb{R}^+$ in $\mathbb{R}^n$.

Then we have $\operatorname{lin}(K) = 0$, $\dim(K) = 1$, and $\operatorname{codim}(K_V) = n - 1$. The proper cone-space pair K_{SP} that we obtain is $(\mathbb{R}^+, \mathbb{R})$, so by Theorem 3,

$$\beta(K_V) = n^2 - n + 1.$$

Reduction formula

Example 6. Suppose that K_V is proper.

Then $S = P = V$, so $K_{SP} = K_V$ and both $\operatorname{lin}(K) = \operatorname{codim}(K_V) = 0$.

Theorem 3 simply reduces to

$$\beta(K_V) = \beta(K_{SP}).$$

Reduction formula

Theorem 3 provides a shortcut for computing the Lyapunov rank of an improper cone.

Input: A cone-space pair K_V.
Output: The Lyapunov rank of K_V.

$$\beta \leftarrow 0$$
$$n \leftarrow \dim(V)$$
$$m \leftarrow \dim(K)$$
$$l \leftarrow \operatorname{lin}(K)$$

Reduction formula

if $m < n$ **then**

$K_V \leftarrow \textsc{restrict}\left(K_V, \operatorname{span}\left(K_V\right)\right)$

$\beta \leftarrow \beta + (n - m)\, n$ ▷ Lemma 1

end if

if $l > 0$ **then**

$K_V \leftarrow \textsc{restrict}\left(K_V, \operatorname{span}\left(K_V^*\right)\right)$

$\beta \leftarrow \beta + lm$ ▷ Lemma 2

end if

return $\beta + \left|\textsc{ll}\left(K_V\right)\right|$ ▷ K_V is proper here

Reduction formula

And when K_V is polyhedral, we can run it.

```
sage: K = random_cone()
sage: K
12-d cone in 34-d lattice N
sage: timeit('len(K.LL())')
5 loops, best of 3: 10.8 s per loop
sage: timeit('K.lyapunov_rank()')
5 loops, best of 3: 289 ms per loop
```

Reduction formula

The reduction formula can be viewed from another perspective.

Theorem. Let K_W be a cone-space pair and $\dim(V) > \dim(W)$. Then K_V is perfect.

Proof. By construction $\operatorname{codim}(K_V) \geq 1$, so Theorem 3 gives $\beta(K_V) \geq \dim(V)$. $\square$

Corollary.

Adding a slack variable to an optimization problem makes the underlying cone perfect.

(but uselessly so)

References

[1] G. Rudolf, N. Noyan, D. Papp, and F. Alizadeh. Bilinear optimality constraints for the cone of positive polynomials. Mathematical Programming, Series B, 129 (2011) 5–31.

[2] M. Orlitzky. The Lyapunov rank of an improper cone (preprint).