Topological Groups in Optimization

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Motivation

Our primary interest in topological groups is to study *Lie groups* (which are topological groups). The Lie group that we are familiar with is $Aut(K)$, the automorphism group of a cone $K \subseteq \mathbb{R}^n$.

Every Lie group has an associated Lie algebra, and the dimension of the Lie algebra associated with $Aut(K)$ is the Lyapunov rank [1] of K.

Motivation

Definition. A *topological group* is a tuple $(G, \mu, \iota, e, \mathcal{T})$ where (G, μ, ι, e) is a group, $(G, \mathcal{T})$ is a topological space, and μ, ι are continuous.

So we should begin by introducing groups and topological spaces.

Groups

Definition. A *group* is a tuple (G, μ, ι, e) where G is a set, μ is associative "multiplication,"

$$\mu : G \times G \to G$$
$$\mu(a, b) = ab$$

and ι is "inverse" on the set:

$$\iota : G \to G$$
$$\iota(a) = a^{-1}$$

Groups

The element e is called the *identity element* of the group, and satisfies $\mu(a, e) = \mu(e, a) = a$ for all a in G.

The explicit function application of μ and ι is laborious in group theory, but makes things clearer when we begin talking about continuity and function composition.

Example. The set of real numbers under addition:

$$\begin{aligned} G &= \mathbb{R} \\ \mu &= \text{plus},\ (a,b) \mapsto a+b \\ \iota &= \text{negate},\ a \mapsto -a \\ e &= 0 \end{aligned}$$

Groups

Example. The set of nonzero real numbers under multiplication:

$$\begin{aligned} G &= \mathbb{R} \setminus \{0\} \\ \mu &= \text{times},\ (a,b) \mapsto a \cdot b \\ \iota &= \text{reciprocal},\ a \mapsto \frac{1}{a} \\ e &= 1 \end{aligned}$$

Groups

Example. The real general linear group $GL_n(\mathbb{R})$ in n dimensions,

$$\begin{aligned} G &= \left\{A \in \mathbb{R}^{n \times n},\ \det(A) \neq 0\right\} \\ \mu &= \text{matrix multiplication},\ (A, B) \mapsto AB \\ \iota &= \text{matrix inverse},\ a \mapsto A^{-1} \\ e &= I \end{aligned}$$

Continuity

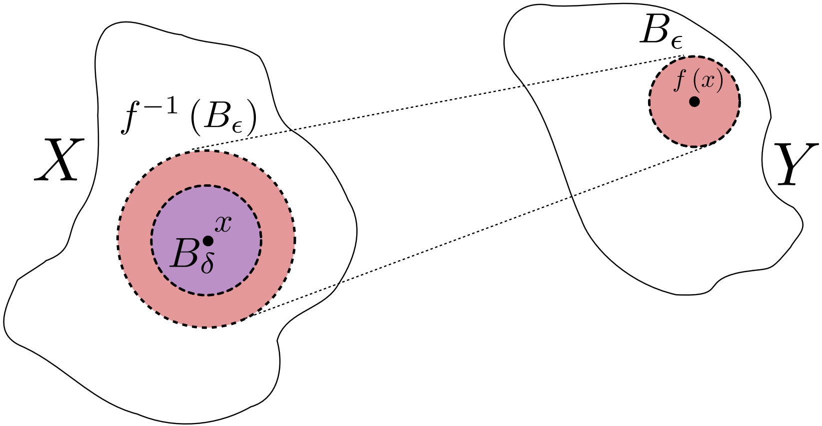
In a metric space, we have the epsilon-delta notion of continuity:

$$f : X \to Y \text{ is continuous at } x \in X$$

$$\Updownarrow$$

$$\forall B_{\epsilon}\left(f\left(x\right)\right), \ \exists B_{\delta}\left(x\right) \subseteq f^{-1}\left(B_{\epsilon}\left(f\left(x\right)\right)\right)$$

Continuity

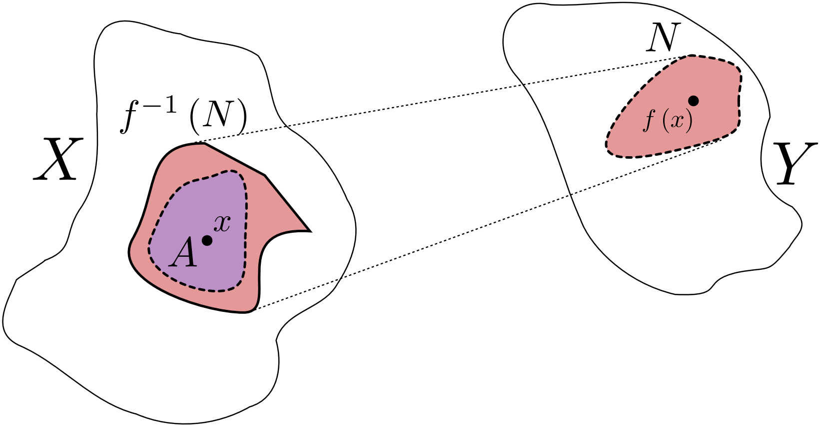


Continuity

We also have the equivalence: $f : X \to Y$ is (epsilon-delta) continuous at x if and only if $f^{-1}(N)$ is a neighborhood of x for every neighborhood N of $f(x)$.

Definition. A *neighborhood N of x* is any set containing an open set $A \ni x$. That is, any set N where $\{x\} \subseteq A \subseteq N$ and A is open. They're used like ϵ-balls, but they can be weirdly-shaped.

Continuity



Continuity

This characterization extends to functions continuous **on** X. Since there is no mention of the underlying metric, this gives us a definition of continuity that works in a more general space:

Definition. A function $f : X \to Y$ is said to be continuous if $f^{-1}(A)$ is open in X for every subset $A \subseteq Y$ open in Y.

Topology

Definition. A *topology* $\mathcal{T}$ *on* X is a collection of subsets of X, which we call "open" by convention. The members of $\mathcal{T}$ must satisfy three criteria:

1. $X, \emptyset \in \mathcal{T}$
2. If $S \subseteq \mathcal{T}$, then $\left(\bigcup_{A \in S} A \right) \in \mathcal{T}$
3. If $S \subseteq \mathcal{T}$ is *finite*, then $\left(\bigcap_{A \in S} A \right) \in \mathcal{T}$

Remark. To show off at parties, point out that the first criterion is technically redundant. The empty set is a (finite) subset of any set, and

$$\mathcal{T} \ni \bigcup_{A \in \emptyset} A = \emptyset$$

$$\mathcal{T} \ni \bigcap_{A \in \emptyset} A = X$$

gives $X, \emptyset \in \mathcal{T}$ from criteria #2 and #3.

Topology

Definition. A *topological space* is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is a topology on X.

Any metric space gives rise to a topological space: let $\mathcal{T}$ be the collection of open sets in the metric space (unions of open ϵ-balls).

The reverse is not true.

Example (indiscrete topology).

$$X = \text{any set}$$
$$\mathcal{T} = \{X, \emptyset\}$$

There's only one possible union and intersection we can form from members of $\mathcal{T}$, and they're both back in $\mathcal{T}$.

There is no associated metric space.

Example (discrete topology).

$$X = \text{any set}$$
$$\mathcal{T} = 2^X$$

Clearly everything we need to be in $\mathcal{T}$ from criteria #1, #2, and #3 is in there, because *everything* is in $\mathcal{T}$.

Example. The set of real numbers with the usual open sets:

$$X = \mathbb{R}$$
$$\mathcal{T} = \text{unions of open intervals}$$

Our three criteria in this case follow from basic properties of open intervals in $\mathbb{R}$.

Example. The set of nonzero real numbers:

$$X = \mathbb{R} \setminus \{0\}$$
$$\mathcal{T} = \bigcup_{i \in I} \left[(a_i, b_i) \setminus \{0\}\right]$$

This is an example of a subspace topology; our X here is a subset of $\mathbb{R}$ (with zero removed), and $\mathcal{T}$ consists of the same sets as in the previous example, except with $\{0\}$ removed from each open interval.

Example. The real general linear group $GL_n(\mathbb{R})$ in n dimensions,

$$X = \left\{A \in \mathbb{R}^{n\times n},\ \det(A) \neq 0\right\}$$
$$\mathcal{T} = \text{the } \|\cdot\| - \text{open sets in } GL_n(\mathbb{R})$$

We have a norm (also a metric) for matrices. $(GL_n(\mathbb{R}), \|\cdot\|)$ is thus a metric subspace of $(M_n(\mathbb{R}), \|\cdot\|)$, and we can use the collection of open sets from the metric space as our $\mathcal{T}$.

Topology

The open-cover definition of compactness uses only the notion of open sets; therefore we have:

Definition. A set if *compact* in a topological space if it is open-cover compact. That is, if every open cover of the given set has a finite subcover.

Beware, some properties of compact sets in metric spaces do not translate!

Example (a set which is compact but not closed).

$$X = \{a, b, c\}$$
$$\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$$

The set $\{a\}$ is compact: all open covers are finite. But it is not closed: $\{b, c\} \notin \mathcal{T}$.

There are a few special types of topological spaces; they have properties that prevent them from being "too weird." The first is,

Definition (T_1 space). A topological space is said to be T_1 if every singleton set is closed in it.

In our previous example, $\{a\}$ was not closed so the space is not T_1.

Definition (Hausdorff space). A topological space is said to be "Hausdorff" (or T_2) if it is T_1 and every $x \neq y$ in the space can be covered by open sets $U \ni x$ and $V \ni y$ that do not intersect; i.e. $U \cap V = \emptyset$.

The space previous example was not T_1, so it is not Hausdorff. Note that all metric spaces are Hausdorff. This property is what is required for compact sets to be closed.

To see this, let $A \subseteq X$ be compact and fix $y \in A^c$. For each pair $\{(x, y) : x \in A\}$ pair we can find sets $U_x \ni y$ and $V_x \ni x$. The V_x cover A, so they have a finite subcover. Keep the corresponding U_x which are finite in number.

Now if we take the (finite!) intersection of the U_x, we get an open set containing y that is disjoint from A. Do this for all $y \in A^c$, and take the union to show that A^c is open. Thus A is closed.

Topology

Definition (Regular space). A topological space is said to be "regular" (or T_3) if it is T_1 and every $x \notin Y$ (where Y is a closed set) in the space can be covered by open sets $U \ni x$ and $V \supseteq Y$ that do not intersect; i.e. $U \cap V = \emptyset$.

This is similar to a Hausdorff space, except the singleton set (point) $\{y\}$ has been replaced with a closed set Y.

Topological Groups

By now this definition should make more sense:

Definition. A *topological group* is a tuple $(G, \mu, \iota, e, \mathcal{T})$ where (G, μ, ι, e) is a group and $(G, \mathcal{T})$ is a topological space.

Furthermore, the group multiplication μ and the group inverse ι are continuous *with respect to the topology $\mathcal{T}$ on G.*

Topological Groups

By definition, the map $x \mapsto \mu(g, x) = gx$ is continuous. The composition of two continuous maps is again continuous, so,

$$x \mapsto \mu(\iota(g), x) = g^{-1}x$$

is also continuous. Thus, multiplication on the left/right is a homeomorphism. Inversion is also obviously a homeomorphism.

Topological Groups

Homeomorphisms preserve open and closed sets, therefore we have,

Corollary.

$$\begin{aligned} H \subseteq G \text{ is open} &\iff gH \text{ is open} \\ &\iff Hg \text{ is open} \\ &\iff H^{-1} \text{ is open.} \end{aligned}$$

Topological Groups

All of our group examples can be thought of as topological groups, since they derive a topology from their metrics:

- $\mathbb{R}$, plus, negate
- $\mathbb{R} \setminus \{0\}$, times, reciprocal
- $GL_n(\mathbb{R})$, matrix multiplication, inverse

Example. Any group (G, μ, ι, e) can be made into a topological group $(G, \mu, \iota, e, \mathcal{T})$ via the discrete topology, $\mathcal{T} = 2^G$.

This is cheating, but μ and ι are automatically continuous because *everything* is in $\mathcal{T}$ when you consider their preimages.

Topological Groups

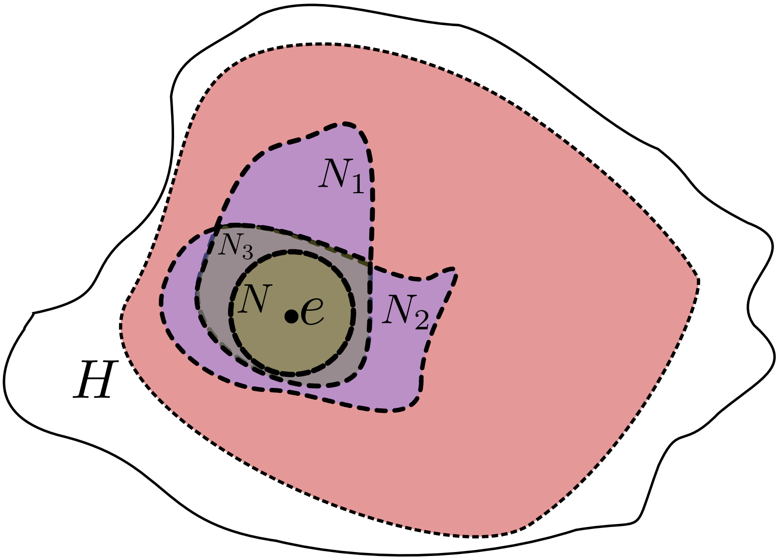
Example. A Lie group is a group (G, μ, ι, e) where G is a differentiable manifold and μ, ι are compatible with the differential structure on G. In particular μ and ι are smooth operations, and are thus continuous. So every Lie group is a topological group.

Example. $Aut\,(K)$, the automorphism group of a proper cone K, is a topological group (a subgroup of $GL_n\,(\mathbb{R})$).

Topological Groups

Proposition. Every neighborhood H of e contains a neighborhood N of e such that $N = N^{-1}$ and $NN \subseteq H$.

To understand the proof, think of $G = \mathbb{R} \setminus \{0\}$ and let $1 \pm \delta$ be a given δ-ball. Can you find an ϵ-ball $B = 1 \pm \epsilon$ around 1 such that $BB = 1 \pm 2\epsilon \pm \epsilon^2$ is contained within $1 \pm \delta$? Sure, easy. The idea is the same.



N_1
N_3
N
e
N_2
H

Proof.

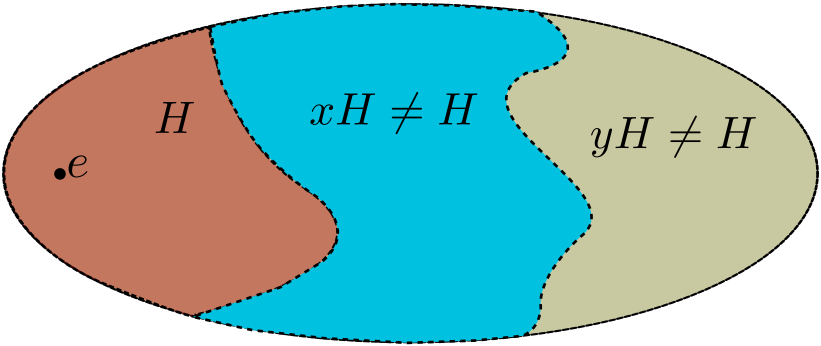
Without loss of generality let H be open (otherwise, take its interior). Then $\mu^{-1}(H)$ is open by continuity of μ, and gives us two open neighborhoods N_1 and N_2 of e such that $\mu(N_1, N_2) \subseteq H$. Intersect the two to get N_3 which is another (smaller) neighborhood of e. Finally take $N = N_3 \cap N_3^{-1}$ to make it symmetric. $\square$

Proposition. Any open subgroup H of G is closed as well.

Proof.

Any coset xH of H is open by continuity, but the cosets of H partition G. So the complement of H in G is just the union of the non-H cosets, and they're all open, too. Therefore, H^c is open and H is closed. $\square$

Topological Groups

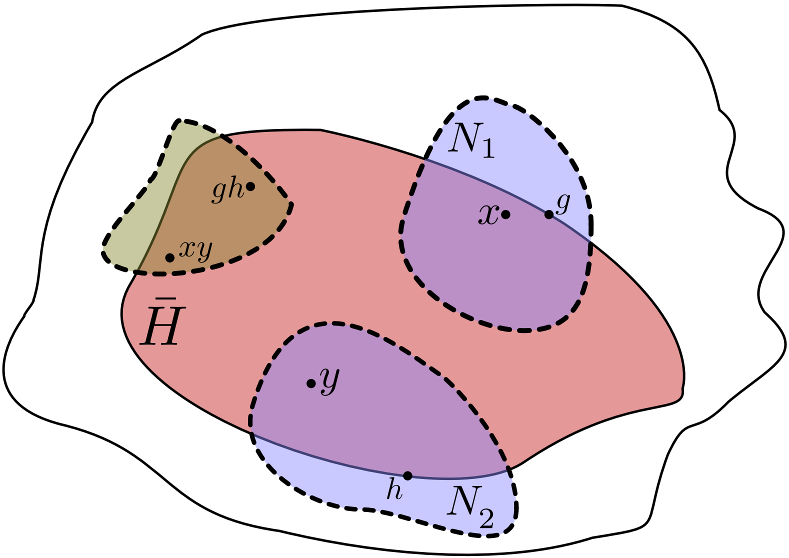


Proposition. If H is a subgroup of G, then $\bar{H}$ is also a subgroup of G.

Proof.

Let $gh \in \bar{H}$; we will show that every open neighborhood N of gh contains an element of H. By continuity, $\mu^{-1}(N) = N_1 \times N_2$ are open neighborhoods of g, h. But $g, h \in \bar{H}$ so N_1 and N_2 contain other points $x, y \in H$ respectively. Since H is a group, we have $xy \in N$ and $xy \in H$. $\square$

Topological Groups



Topological Groups

Let $H \subseteq G$ be a subgroup of G. From group theory, we know we can define a quotient G/H with

$$p : G \to G/H$$
$$p(g) = [g]$$

as its projection map. The function p takes an element $g \in G$ to its equivalence class in G/H.

We can define a topology $\mathcal{Q}$ on the quotient G/H in a natural way.

Definition. The *quotient topology* $\mathcal{Q}$ is the natural topology defined on G/H where $X \subseteq G/H$ is open if and only if $p^{-1}(X)$ is open in G.

In other words, $X \in \mathcal{Q} \iff p^{-1}(X) \in \mathcal{T}$.

Note: we can construct the quotient topology even when H is not normal; i.e. when G/H is not a group!

The quotient topology is "natural" because it makes the projection map p continuous by definition. It also happens to be an open map.

Proof.

Let $X \subseteq G$ be open; then $p(X)$ is open in G/H if $p^{-1}(p(X))$ is open in G by definition. But,

$$\begin{aligned} p^{-1}(p(X)) &= \{x \in G : p(x) \in p(X)\} \\ &= \{xH : x \in X\} \\ &= XH \\ &= \bigcup_{h \in H} Xh \end{aligned}$$

where each Xh is open. □

If H is compact, then p is also a closed map.

Proof.

The proof is identical, except that when we reach the product $XH = \cup_{h \in H} Xh$, the union is not necessarily closed. But it can be shown to be closed when H (or X, in general) is compact. $\square$

Topological Groups

Proposition. If H is normal, then G/H is a topological group

Proof. This is "obvious," but we need to show that multiplication and inverse are continuous in G/H. To do this, note that p is an open map and both $\mu(g, \cdot)$ and ι are continuous. Moreover, H is normal, so $(gx)H = gxHx^{-1}xH = (gH)(xH)$.

Topological Groups

Proof (continued). Therefore, the following diagrams commute:

$$
\begin{array}{ccc}
G & \xrightarrow{\mu(g,\cdot)} & G \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\
G/H & \xrightarrow{\mu(p(g),\cdot)} & G/H
\end{array}
\qquad
\begin{array}{ccc}
G & \xrightarrow{\iota} & G \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\
G/H & \xrightarrow{\iota'} & G/H
\end{array}
$$

Topological Groups

These ideas extend to Lie groups.

Theorem (Hilgert & Neeb, 9.3.7). Let (G, μ, ι, e) be a Lie group, and let $H \subseteq G$ be a closed subgroup of G. Then H is a Lie group.

Theorem (Hilgert & Neeb, 11.1.5). If in addition H is normal, then G/H is a Lie group.

Topological Groups

Definition. A topological space $(X, \mathcal{T})$ is called *locally compact* if every point $x \in X$ has a compact neighborhood $N \ni x$.

Lemma. If $(X, \mathcal{T})$ is locally-compact and Hausdorff, then it is also regular, and any neighborhood of $x \in X$ contains a compact neighborhood of x.

Proof (omitted). Purely topological.

Topological Groups

Lemma. Let G be a Hausdorff topological group and let H be a locally-compact subgroup of G. Then H is closed.

Without the fact that H is a subgroup, it would need to be *globally*-compact to be necessarily closed.

Corollary. Locally-compact subgroups of Hausdorff Lie groups are Lie groups.

Topological Groups

Definition. A topological space $(X, \mathcal{T})$ is said to be homogeneous if for any $x, y \in X$, there is a homeomorphism sending x to y.

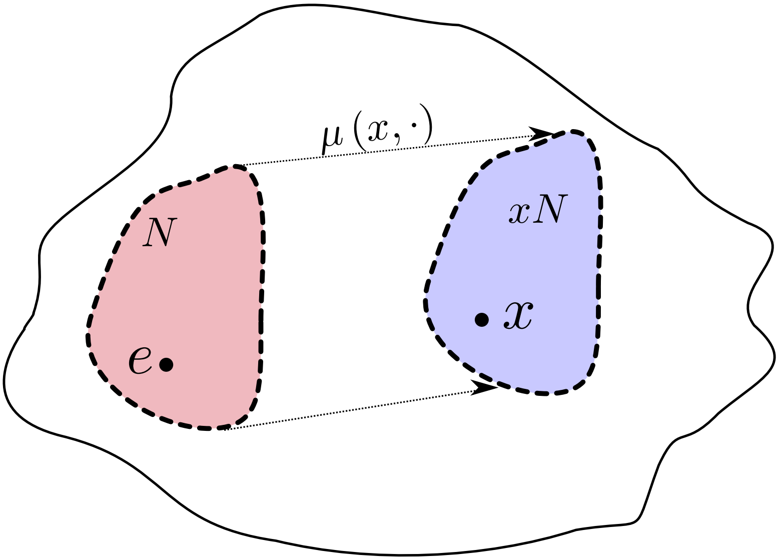
Every topological group is homogeneous, since the map $f(g) = gx^{-1}y$ is a homeomorphism and $f(x) = xx^{-1}y = y$.

Topological Groups

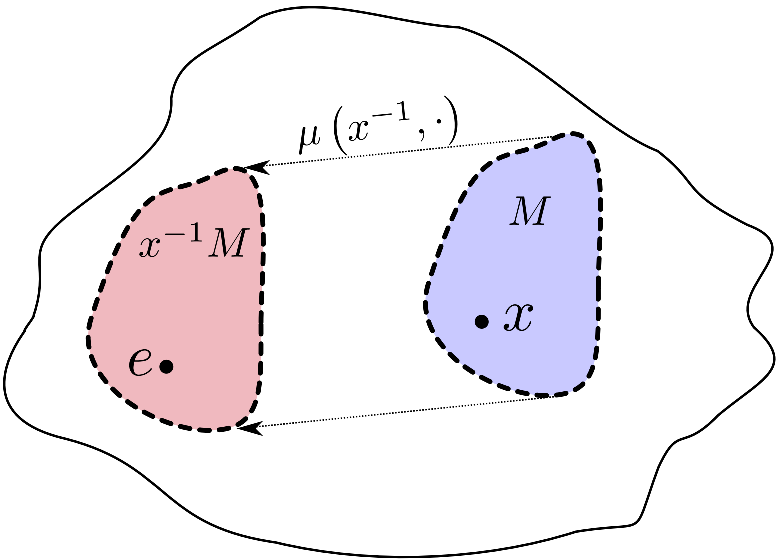
If two sets in a topological space are connected by a homeomorphism, they are "essentially the same." This means that we can study an entire topological group by looking at neighborhoods of the identity.

Example. Every neighborhood in a topological group is a translation of a neighborhood of the identity. This is used heavily in proofs.

Topological Groups



Topological Groups



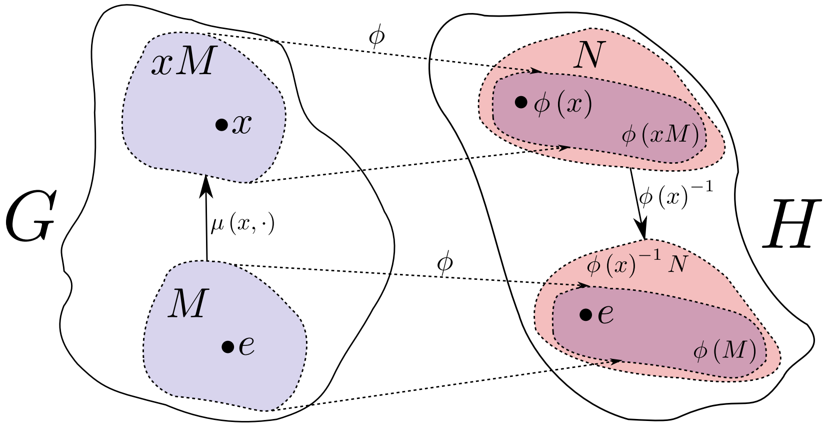
Topological Groups

Lemma. A topological group homomorphism $\phi : G \to H$ is continuous if it is continuous at the identity.

Proof. Suppose ϕ is continuous at $e \in G$, and let $N \ni \phi(x) \in H$ be given. By continuity, we can find an $M \ni e$ such that $\phi(M) \subseteq \phi(x)^{-1} N$. But then,

$$\phi(x)\,\phi(M) = \phi(xM) \subseteq \phi(x)\,\phi(x)^{-1} N = N.$$

Since ϕ is a homomorphism, "going up then right" is the same as "going right then up."



Definition. A topological space $(X, \mathcal{T})$ is said to be *connected* if it has no nonempty proper clopen subsets. This is equivalent to saying that any two nonempty open subsets $A \cup B = X$ have nonempty intersection.

Many familiar properties of connectedness (for example, it is preserved under a continuous function) transfer from metric spaces.

Topological Groups

Definition. A maximal connected subset of X is called a *connected component*.

Definition. The space $(X, \mathcal{T})$ is *totally disconnected* if each singleton set is its own connected component.

Proposition. If A is connected, then so is $\bar{A}$.

Proof (contrapositive). Suppose $\bar{A}$ is disconnected; i.e. $\bar{A} = B \cup B^c$ where B is clopen in the topology *relative to* $\bar{A}$. Then we can write A as $A \cap \bar{A} = (A \cap B) \cup (A \cap B^c)$. Now both $(A \cap B)$ and $(A \cap B^c)$ are clopen in the topology relative to A, and at least one is nonempty, so A is disconnected.

Topological Groups

Let G° represent the connected component of the identity in G.

Lemma. G is totally disconnected if and only if $G^\circ = \{e\}$.

Lemma. Every connected component in G is of the form xG° for some $x \in G$.

Proof. Homogeneity.

Lemma. G° is a closed, normal subgroup of G.

Proof.

If $g \in G^\circ$, then by continuity, $g^{-1}G^\circ$, $(G^\circ)^{-1}$, and $xG^\circ x^{-1}$ are all connected and each contains the identity. G° is the largest such set, so all three must be contained in G°. Therefore G° is a group and it is normal. Connected components are always closed. $\square$

Topological Groups

Lemma. Connected matrix Lie groups such as $Aut\,(K)^{\circ}$ are path-connected.

Proof.

Lie groups are smooth manifolds, and are therefore locally path-connected. If the entire group is connected, a global path can be constructed by stitching together local ones. □

Topological Groups

An example of a theorem involving this concept can be found in *Analysis on Symmetric Cones* [2]:

Theorem (Faraut & Korányi, III.2.1). Let K be the cone of squares in a Euclidean Jordan algebra V, and let $V^{\times}$ be the set of units (invertible elements) in V. Then $\operatorname{int}(K)$ is the connected component of the identity in $V^{\times}$.

References I

[1] M.S. Gowda and J. Tao. On the bilinearity rank of a proper cone and Lyapunov-like transformations. Mathematical Programming, 147 (2014) 155-170.

[2] J. Faraut and A. Korányi. Analysis on Symmetric Cones. Oxford University Press, New York, 1994.

[3] J. Hilgert and K-H. Neeb. Structure and Geometry of Lie Groups. Springer, 2012.

References II

[4] G. McCarty. Topology: An Introduction with Application to Topological Groups. Dover, 1988.

[5] R. Vinroot. Topological Groups. Retrieved from `http://www.math.wm.edu/~vinroot/PadicGroups/topgroups.pdf`.