Continuity of the conic hull

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Abstract

In a real Hilbert space V, the conic hull of $G \subseteq V$ is the set $\operatorname{cone}(G)$ consisting of all nonnegative linear combinations of elements of G. Many optimization problems are sensitive to the changes in $\operatorname{cone}(G)$ that result from changes in G itself. Motivated by one such problem, we derive necessary and sufficient conditions for the continuity of the conic hull.

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1 Introduction

Our goal today is to characterize the continuity of the conic hull. We set the stage with a bit of jargon. Everything takes place in a finite-dimensional real Hilbert (henceforth: *Euclidean*) space that we'll call V and whose unit sphere is denoted by $\mathbb{S}(V)$.

A *cone* is a nonempty subset K of V such that $\alpha K \subseteq K$ for all $\alpha \geq 0$. A *closed convex cone* is a cone that is closed and convex as a subset of V. The *conic hull* of a subset $G \subseteq V$ is a convex cone,

$$\operatorname{cone}(G) \coloneqq \left\{ \sum_{i=1}^{m} \alpha_i g_i \;\middle|\; g_i \in G, \; \alpha_i \geq 0, \; m \in \mathbb{N} \right\}.$$

On other days, the conic hull might also be called the *positive span* [2] or the *positive hull* [7, 9, 12]. If $\operatorname{cone}(G) = K$, then G is said to *generate* K and the elements of G are *generators* of K. If G is finite, then K is closed [9]. The *lineality space* of a convex cone K is $\operatorname{linspace}(K) \coloneqq -K \cap K$ and K is *pointed* if $\operatorname{linspace}(K) = \{0\}$.

Question. When do small changes in G produce small changes in $\operatorname{cone}(G)$?

1.1 Some motivation

Iusem and Seeger [6] define the maximal angle within a nontrivial closed convex cone K to be

$$\theta_{\max}(K) \coloneqq \max\left(\{ \arccos \langle x, y \rangle \mid x, y \in K \cap \mathbb{S}(V) \} \right).$$

The underlying computation here is a conic version of a familiar optimization problem, that of finding a pair of points within a compact set that are at maximal distance from one another. Said pairs are called antipodal, and the distance between them is known as the diameter of the set. Gourion and Seeger found an algorithm to compute the maximal angle [3], and their result was later enriched by Seeger and Sossa with support for multiple cones [10]. The maximal angle between *two* nontrivial closed convex cones P and Q is defined to be

$$\Theta(P,Q) \coloneqq \max\left(\{\arccos\langle x,y\rangle \mid x \in P \cap \mathbb{S}(V), y \in Q \cap \mathbb{S}(V)\}\right),$$

and may be seen as a generalization of the principal angle between linear subspaces [10]. Of course, $\theta_{\max}(K) = \Theta(K,K)$.

From any set of generators, a minimal conically-independent subset can be chosen [2] and scaled to unit norm without affecting the cone; the Seeger-Sossa algorithm for computing $\Theta(P,Q) = \Theta(\text{cone}(G), \text{cone}(H))$ depends explicitly on two such sets G and H that generate P and Q, respectively. Seeger and Sossa furthermore demonstrate that the map $(P,Q) \mapsto \Theta(P,Q)$ is continuous. We are thereby compelled to ask whether or not the conic hull is continuous, so that the whole expression might depend continuously on G and H. Simply put, it'd be nice to know if the algorithms to compute maximal angles have any hope of being stable.

But that's not all: the Seeger-Sossa algorithm involves the solution of several generalized eigenvalue problems, each with matrices formed from the generating sets G and H. Orlitzky noticed that, should they arise, eigenspaces of dimension two or greater can subvert the algorithm [8]. In numerical experiments this situation has been avoided, one supposes, because the eigenspaces tend to dimension one in floating point. This prompts Orlitzky to ask if small perturbations of G and H can be used to eliminate the unwelcome eigenspaces, and if so, whether or not those perturbations will significantly change the answer.

1.2 Other approaches

Walkup and Wets were likely the first to investigate the continuity of the conic hull [12]. They first exhibit continuity where the resulting cone is pointed, and then use that fact to derive two conditions that together imply continuity more generally[1]. Take heed that Walkup and Wets consider the domain of the conic hull to be the space of real m-by-n matrices endowed with the "long vector" norm from $\mathbb{R}^{mn}$. Their conic hull then consists of all nonnegative linear combinations of the columns of its argument. This representation imposes a fixed order and cardinality upon the columns-*cum*-generators.

Corollary 1 (Walkup-Wets)**.** *If* $\text{cone}(A)$ *is pointed and if none of the columns of* $A \in \mathbb{R}^{m \times n}$ *is zero, then the conic hull is continuous in a neighborhood of* A*.*

Theorem 1 (Walkup-Wets)**.** *Suppose that Z is a subset of $\mathbb{R}^{m \times n}$, that k is an integer, and that for every matrix A in Z,*

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[1] The corollary to the theorem is in fact independent of the theorem and is used in its proof.

1. $\dim\left(\text{linspace}\left(\text{cone}\left(A\right)\right)\right) = k,$

2. *there exists a neighborhood N about A such that if any column A^i of A lies in $\text{linspace}\left(\text{cone}\left(A\right)\right)$, then the corresponding column $\tilde{A}^i$ of any matrix $\tilde{A}$ in $N \cap Z$ lies in $\text{linspace}\left(\text{cone}\left(\tilde{A}\right)\right)$.*

Then the restriction of the conic hull to Z is continuous.

Our qualm with these conditions is that they aren't practical to verify on the fly. If you can find a set where nothing bad happens, and look only inside that set, then ipso facto you should see only good things happen. But proving that you have or can find such a set is often tantamount to the problem you started with, and the computer is ill-equipped to assist.

Luc and Wets revisit the problem [7] armed with a few decades' worth of new technology, the standard reference for which is the opus of Rockafellar and Wets [9]. For contrast, the matrices of Walkup and Wets are replaced by arbitrary sets, and long-vector limits are superseded by Painlevé-Kuratowski outer limits of sets [1, 9].

Theorem 2 (Luc-Wets)**.** *Given a collection $\{W; W^\nu \mid \nu \in \mathbb{N}\}$ of nonempty subsets of $\mathbb{R}^n$,*

$$\limsup_\nu \text{cone}\left(W^\nu\right) \subseteq \text{cone}\left(W\right)$$

under the following hypotheses:

1. *W includes the outer limit of the sets W^ν,*

2. *$\text{cone}\left(W\right)$ includes the horizon outer limit of the sets W^ν,*

3. *$\text{cone}\left(W\right)$ includes the core outer limit of the sets W^ν,*

4. *if $t \in \text{linspace}\left(\text{cone}\left(W\right)\right)$ is nonzero and a cluster point of the sequence $\left\{\frac{t^\nu}{\|t^\nu\|} \;\middle|\; \nu \in \mathbb{N}, t^\nu \in \text{cone}\left(W^\nu\right) \setminus \{0\}\right\}$, then $t^\nu \in \text{linspace}\left(\text{cone}\left(W^\nu\right)\right)$ for ν large enough, and $\text{linspace}\left(\text{cone}\left(W\right)\right) \supseteq \limsup_\nu \text{linspace}\left(\text{cone}\left(W^\nu\right)\right)$.*

On the one hand, the added generality has several nice consequences, the most magnanimous being the absence of any real restrictions on the generating sets. On the other, greater generality is no boon for usability. There are now four hypotheses that together imply outer-semicontinuity but that remain challenging to verify mechanically. Conspicuously, neither theorem is apt to contend with numerical noise.

1.3 Our approach

The subtext of Theorems 1 and 2 is that discontinuity is born of lineality spaces, but neither ventures a necessary condition. Instead they allow the user to promise that nothing goes awry in the lineality space, salvaging continuity if he is able to do so. We challenge this with a simplification, refusing to focus only

on a subset of the domain or on certain sequences[2]. After fixing its domain (which we do in a moment), we ask for epsilon-delta continuity of the conic hull in the usual metric sense at a point. This yields a necessary condition as problematic perturbations cannot be hypothesized away.

Towards that end, we borrow the set-based convergence of Luc and Wets, imposing additional restrictions until we have honest-to-goodness metric spaces in both the domain and codomain of the conic hull.

Definition 1 (admissible cones and generating sets)**.** The Pompeiu-Hausdorff distance [1, 9] between two nonempty compact sets $X, Y \subseteq V$ is,

$$\text{haus}\left(X, Y\right) \coloneqq \max\left(\left\{\max_{y \in Y} \text{dist}\left(y, X\right), \max_{x \in X} \text{dist}\left(x, Y\right)\right\}\right),$$

where $\text{dist}\left(x, Y\right) = \min\left(\left\{\|x - y\| \mid y \in Y\right\}\right)$ denotes the usual metric distance from the point x to the set Y in V. For the domain of the conic hull, we take the space of *admissible generating sets* in V,

$$\mathcal{G}\left(V\right) \coloneqq \left\{G \subseteq V \mid G \neq \emptyset,\ G \text{ is finite, and } 0 \notin G\right\},$$

equipped with the Pompeiu-Hausdorff distance as its metric. For the codomain of the conic hull we choose the space of *admissible cones* in V,

$$\mathcal{C}\left(V\right) \coloneqq \left\{K \subseteq V \mid K \text{ is a closed convex cone and } K \neq \{0\}\right\},$$

instilled with the *spherical metric*,

$$\sigma\left(P, Q\right) \coloneqq \text{haus}\left(P \cap \mathbb{S}\left(V\right), Q \cap \mathbb{S}\left(V\right)\right).$$

Convergence in either space is metric [1], and in the case of $\mathcal{C}\left(V\right)$ where the sets are *uniformly* bounded, is equivalent to Painlevé-Kuratowski convergence [9]. Appendix A chronicles the evolution of these definitions. Hereafter we interpret the conic hull as a map from $\mathcal{G}\left(V\right)$ to $\mathcal{C}\left(V\right)$.

The sets in $\mathcal{G}\left(V\right)$ are not typically convex and may therefore contain multiple elements at minimal distance from a given $x \in V$.

Definition 2. The *projection set* of an element $x \in V$ onto $G \in \mathcal{G}\left(V\right)$ is

$$\pi_G\left(x\right) \coloneqq \left\{g \in G \mid \|g - x\| = \text{dist}\left(x, G\right)\right\}.$$

When we need a point in G at minimal distance from x, we will select an arbitrary element from $\pi_G\left(x\right)$.

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[2]This is both honest and untrue, since our domain itself excludes many pesky sets, though in a way that does not compromise the integrity of our results.

2 Towards necessity

We shall ultimately see that the conic hull is continuous at $G \in \mathcal{G}(V)$ if and only if either $\text{cone}(G)$ is pointed or $\text{cone}(G) = V$. We attack each constituent of this statement individually: the sufficiency of being pointed, the sufficiency of being the entire space, and finally, the necessity of at least one of those.

Except at the origin, pointed cones are strictly contained within a single half-space. This fact follows from a venerable separation theorem and appears in many disguises; Exercise 6.22 of Rockafellar and Wets is conveniently clad [9].

Lemma 1. *If K is a closed convex cone in a nontrivial Euclidean space V, then K is pointed if and only if there exists some $q \in \mathbb{S}(V)$ such that $\langle x, q \rangle > 0$ for all nonzero x in K.*

For convenience we have scaled the vector q in this result to have unit norm, but doing so required that V be made nontrivial. In the space $V = \{0\}$, our own results hold vacuously because $\mathcal{G}(V)$ is empty. We will therefore sneakily apply Lemma 1 while assuming implicitly that V is nontrivial. (One does not need any new theorems to understand completely the behavior of the conic hull on the trivial space.)

Iusem and Seeger showed that the set of pointed cones is open within the space of admissible cones [5], and we begin by showing that this result can be pulled back to the generating sets.

Proposition 1. *If V is a Euclidean space, if $G \in \mathcal{G}(V)$, and if $\text{cone}(G)$ is pointed, then there exists a $\delta > 0$ such that $\text{cone}(\tilde{G})$ is pointed whenever $\text{haus}(G, \tilde{G}) < \delta$.*

Proof. If $\text{cone}(G)$ is pointed, then there exists by Lemma 1 some $q \in \mathbb{S}(V)$ such that $\langle g, q \rangle > 0$ for all $g \in G$. Define $\delta := \min(\{\langle g, q \rangle \mid g \in G\})/2$, and suppose $\text{haus}(G, \tilde{G}) < \delta$. Then for each $\tilde{g} \in \tilde{G}$ there is a $g \in \pi_G(\tilde{g})$ with

$$|\langle g, q \rangle - \langle \tilde{g}, q \rangle| \leq \|g - \tilde{g}\| < \delta.$$

But δ, by definition, is no more than half the distance from zero to any $\langle g, q \rangle$, and each $\langle g, q \rangle$ is positive. Thus each $\langle \tilde{g}, q \rangle$ is positive as well. Apply Lemma 1 again to conclude that $\text{cone}(\tilde{G})$ is pointed. $\square$

Corollary 2. *If V is a Euclidean space, then $\{G \in \mathcal{G}(V) \mid \text{cone}(G) \text{ is pointed}\}$ is an open subset of $\mathcal{G}(V)$.*

Proof. A trivial verification in $V = \{0\}$, and Proposition 1 elsewhere. $\square$

We next show that the conic hull is continuous at G when $\text{cone}(G)$ is pointed. This result bears overt similarity to Corollary 1, but is not a consequence of it, owing to the incompatible domains.

Proposition 2. *If V is a Euclidean space, if $G \in \mathcal{G}(V)$, and if $\text{cone}(G)$ is pointed, then for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $\text{haus}(G, \tilde{G}) < \delta$ implies $\sigma(\text{cone}(G), \text{cone}(\tilde{G})) < \varepsilon$.*

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Proof. Let $x = \sum_{i=1}^{m} \alpha_i g_i$ be an arbitrary unit-norm element of $\mathrm{cone}\,(G)$. As $\mathrm{cone}\,(G)$ is pointed, there exists by [Lemma 1](#) some $q \in \mathbb{S}\,(V)$ such that $\langle g, q \rangle > 0$ for all $g \in G$. Define $\gamma := \min\,(\{\langle g, q \rangle \mid g \in G\})$, and note that

$$\gamma \sum_{i=1}^{m} \alpha_i \leq \sum_{i=1}^{m} \alpha_i \langle g_i, q \rangle = \langle x, q \rangle \leq \|x\|\,\|q\| = 1.$$

Thus we have a bound $\sum_{i=1}^{m} \alpha_i \leq \gamma^{-1}$ that is independent of x.

Set $\delta := \varepsilon\gamma/4$, and suppose that $\mathrm{haus}\,(G, \tilde{G}) < \delta$. For each $g \in G$, we may choose one $\tilde{g} \in \pi_{\tilde{G}}\,(g)$ to define $\tilde{x} := \sum_{i=1}^{m} \alpha_i \tilde{g}_i \in \mathrm{cone}\,(\tilde{G})$. Then,

$$\|x - \tilde{x}\| = \left\|\sum_{i=1}^{m} \alpha_i\,(g_i - \tilde{g}_i)\right\| < \delta \sum_{i=1}^{m} \alpha_i \leq \varepsilon/4.$$

While x has unit norm, there is no such guarantee for $\tilde{x}$. But the reverse triangle inequality gives

$$|1 - \|\tilde{x}\|| = |\|x\| - \|\tilde{x}\|| \leq \|x - \tilde{x}\| \leq \varepsilon/4$$

and this leads to

$$\left\|x - \frac{\tilde{x}}{\|\tilde{x}\|}\right\| \leq \|x - \tilde{x}\| + \left\|\tilde{x} - \frac{\tilde{x}}{\|\tilde{x}\|}\right\| \leq \varepsilon/4 + |\|\tilde{x}\| - 1| \leq \varepsilon/2.$$

Thus having $\mathrm{haus}\,(G, \tilde{G}) < \delta$ implies that $\mathrm{dist}\,(x, \mathrm{cone}\,(\tilde{G}) \cap \mathbb{S}\,(V)) < \varepsilon$ for all $x \in \mathrm{cone}\,(G) \cap \mathbb{S}\,(V)$, and it follows that

$$\max\,(\{\mathrm{dist}\,(x, \mathrm{cone}\,(\tilde{G}) \cap \mathbb{S}\,(V)) \mid x \in \mathrm{cone}\,(G) \cap \mathbb{S}\,(V)\}) < \varepsilon$$

whenever $\mathrm{haus}\,(G, \tilde{G}) < \delta$. This is one half of the result.

If necessary, use [Proposition 1](#) to shrink δ until $\mathrm{cone}\,(\tilde{G})$ becomes pointed. Then repeat the above argument with G and $\tilde{G}$ switched. After possibly shrinking δ again, you should conclude that $\mathrm{haus}\,(G, \tilde{G}) < \delta$ implies

$$\max\,(\{\mathrm{dist}\,(\tilde{x}, \mathrm{cone}\,(G) \cap \mathbb{S}\,(V)) \mid \tilde{x} \in \mathrm{cone}\,(\tilde{G}) \cap \mathbb{S}\,(V)\}) < \varepsilon,$$

which is the other half of $\sigma\,(\mathrm{cone}\,(G), \mathrm{cone}\,(\tilde{G})) < \varepsilon$. $\qquad\square$

Attempting to prove the converse, we hit a roadblock: if $\mathrm{cone}\,(G)$ is the entire space, the situation appears stable. This turns out to be an exceptional case that we can prove with assistance from Chandler Davis [2].

Lemma 2 (Davis)**.** *If V is a Euclidean space and if $G \in \mathcal{G}\,(V)$, then $\mathrm{cone}\,(G) = V$ if and only if for all nonzero $x \in V$ there exists a $g \in G$ such that $\langle g, x \rangle > 0$.*

Proposition 3. *If V is a Euclidean space, if $G \in \mathcal{G}\,(V)$, and if $\mathrm{cone}\,(G) = V$, then there exists a $\delta > 0$ such that $\mathrm{haus}\,(G, \tilde{G}) < \delta$ implies $\mathrm{cone}\,(\tilde{G}) = V$.*

Proof. Suppose $\text{cone}\,(G) = V$, and use Lemma 2 to define the function,

$$\gamma : V \to (0, \infty)$$
$$\gamma = x \mapsto \max\left(\{\langle g, x\rangle \mid g \in G\}\right).$$

For an arbitrary $x \in V$, let G_x denote any element of G satisfying $\langle G_x, x\rangle = \gamma(x)$. To see that the function γ is continuous, observe the two inequalities

$$\gamma(x + \Delta x) \leq \max\left(\{\langle g, x\rangle \mid g \in G\}\right) + \max\left(\{\langle g, \Delta x\rangle \mid g \in G\}\right)$$
$$\leq \gamma(x) + \max_{g \in G} \|g\| \, \|\Delta x\|,$$

and

$$\gamma(x + \Delta x) = \max\left(\{\langle g, x\rangle + \langle g, \Delta x\rangle \mid g \in G\}\right)$$
$$\geq \langle G_x, x\rangle + \langle G_x, \Delta x\rangle$$
$$\geq \gamma(x) - \max_{g \in G} \|g\| \, \|\Delta x\|.$$

Thus γ achieves its minimum on $\mathbb{S}(V)$. Define $\gamma_{\min}$ to be that minimum, and let δ be smaller than $\gamma_{\min}$. If $\text{haus}\,(G, \tilde{G}) < \delta$, then, in particular, when $x \in \mathbb{S}(V)$, there exists a $\widetilde{G_x} \in \pi_{\tilde{G}}(G_x)$ with $\left\|G_x - \widetilde{G_x}\right\| < \delta$, and

$$\left|\langle G_x, x\rangle - \left\langle \widetilde{G_x}, x\right\rangle\right| \leq \left\|G_x - \widetilde{G_x}\right\| \|x\| < \delta < \gamma_{\min}.$$

But $\langle G_x, x\rangle \geq \gamma_{\min}$ by the definition of $\gamma_{\min}$, so $\left\langle \widetilde{G_x}, x\right\rangle > 0$ for all $x \in \mathbb{S}(V)$. Scale Lemma 2 to $\mathbb{S}(V)$ and apply it once more to see that $\text{cone}\,(\tilde{G}) = V$. $\square$

Corollary 3. *If V is a Euclidean space, then $\{G \in \mathcal{G}(V) \mid \text{cone}\,(G) = V\}$ is an open subset of $\mathcal{G}(V)$.*

Proof. A trivial verification in $V = \{0\}$, and Proposition 3 elsewhere. $\square$

With that exception out of the way, the necessary condition is within reach. To prove it, we'll decompose our closed convex cone into a vector subspace and a pointed cone. This lucrative result appears, for example, as Stoer and Witzgall's Theorem 2.10.5 [11].

Lemma 3. *Every convex cone K in a Euclidean space has an orthogonal direct sum decomposition,*

$$K = \text{linspace}\,(K) \oplus K \cap \text{linspace}\,(K)^{\perp},$$

wherein $K \cap \text{linspace}\,(K)^{\perp}$ is a pointed convex cone.

When G generates K, the subset of G that lies in the lineality space of K is called its *lineal part*. Wets and Witzgall [13] showed that the lineal part of G corresponds exactly to $\text{linspace}\,(K)$ in the direct sum decomposition of K. It follows that if $\text{linspace}\,(K) \neq \{0\}$, then the origin is a nontrivial (not all coefficients zero) conic combination of the elements of G itself.

Lemma 4 (Wets-Witzgall). *If V is a Euclidean space, if $G \in \mathcal{G}(V)$, and if* $\text{cone}(G) = K$, *then* $\text{cone}(G \cap \text{linspace}(K)) = \text{linspace}(K)$.

Corollary 4. *If V is a Euclidean space, if $G \in \mathcal{G}(V)$, and if $\text{cone}(G)$ is not pointed, then the origin is a nontrivial conic combination of the elements of G.*

We're now ready to prove that the conic hull is *not* continuous at G when $\text{cone}(G)$ is neither pointed nor the entire space. The direct sum decomposition allows us to choose a direction that points away from both the lineality space of $\text{cone}(G)$ and from its pointed component. Perturbing the elements of G in that direction will result, after taking the conic hull, in a unit-norm vector that is bounded away from any unit-norm element of $\text{cone}(G)$ itself.

Proposition 4. *If V is a Euclidean space, if $G \in \mathcal{G}(V)$, and if $\text{cone}(G)$ is neither V nor pointed, then for all $\delta > 0$ there exists a $\tilde{G} \in \mathcal{G}(V)$ with* $\text{haus}(G, \tilde{G}) < \delta$ *and* $\sigma(\text{cone}(G), \text{cone}(\tilde{G})) \geq 1$.

Proof. Let $K := \text{cone}(G)$. If $K \neq V$, then $\text{linspace}(K)^\perp$ is nontrivial; and as regards Lemma 3, $K \cap \text{linspace}(K)^\perp$ is pointed not only in V but also as a subset of $\text{linspace}(K)^\perp$. Lemma 1 therefore produces a unit-norm $q \in \text{linspace}(K)^\perp$ such that $\langle y, q \rangle > 0$ for all nonzero $y \in K \cap \text{linspace}(K)^\perp$.

Define $\tilde{G} := \{g - \delta q/2 \mid g \in G\}$ assuming without loss of generality that δ is small enough to ensure $0 \notin \tilde{G}$. Next use Corollary 4 to write the origin as a nontrivial conic combination, $0 = \sum_{i=1}^{m} \alpha_i g_i$, of the elements of G. Then

$$-q = \left(\frac{2}{\delta \sum_{i=1}^{m} \alpha_i} \right) \sum_{i=1}^{m} \alpha_i (g_i - \delta q/2) \in \text{cone}(\tilde{G}),$$

so we have both $\text{haus}(G, \tilde{G}) < \delta$ and $-q \in \text{cone}(\tilde{G}) \cap \mathbb{S}(V)$.

Take any $z \in K \cap \mathbb{S}(V)$ and use Lemma 3 to write $z = x + y$ with $x \in \text{linspace}(K)$ and $y \in K \cap \text{linspace}(K)^\perp$. Then,

$$\|-q - (x + y)\|^2 = \|q\|^2 + 2 \langle q, x + y \rangle + \|x + y\|^2 = 2 + 2 \langle q, y \rangle \geq 2.$$

Finally, take square roots to see that the distance between $-q$ and z is at least unity; thus $\sigma(\text{cone}(G), \text{cone}(\tilde{G})) \geq 1$. $\qquad\square$

3   Wrapping up

The quantity $\dim(\text{linspace}(K))$ is the *lineality* of K, and there are well-known algorithms to find it. To produce our main result, we combine Propositions 2, 3, and 4, and rephrase them in terms of lineality.

Theorem 3. *If V is a Euclidean space and if $G \in \mathcal{G}(V)$, then the conic hull, as a map from $\mathcal{G}(V)$ to $\mathcal{C}(V)$, is continuous at G if and only if the lineality of $\text{cone}(G)$ is either zero or $\dim(V)$.*

Back in the land of maximal angles, this success can be parlayed into a sufficient condition for the two-cone problem. Some gymnastics are required to circumvent a technical obstacle: in Seeger and Sossa's proof that the maximal angle map $(P, Q) \mapsto \Theta(P, Q)$ is continuous [10], the arguments P and Q are prohibited from being the ambient space V. The restriction isn't essential to their line of reasoning, but it's easier to check the additional case where $P = V$ or $Q = V$ than it is to verify that statement.

Corollary 5. *Suppose V is a Euclidean space and that $G, H \in \mathcal{G}(V)$. If both* cone (G) *and* cone (H) *are pointed, or if either equals V, then the map $(X, Y) \mapsto \Theta(\text{cone}(X), \text{cone}(Y))$ from $\mathcal{G}(V)^2$ to $[0, \pi]$ is continuous at (G, H).*

Proof. If $P := \text{cone}(G)$ and $Q := \text{cone}(H)$ are both pointed, then Proposition 1 shows that we are in no danger of generating V near G or H. And in that case, we simply cite the continuity of the composition.

In the other, we may assume without apology [2] that $P = V$. Then $\Theta(P, Q) = \pi$, since there exists at least one nonzero $x \in Q$, and $-x \in P$ makes an angle of π with it. If $\tilde{G}$ is close enough to G, then cone $(\tilde{G})$ will still be V by Proposition 3, leaving the maximal angle unchanged. $\square$

The necessary condition does not propagate as easily, however, because you can have continuity around a pair of sets that both generate non-pointed cones.

Example 1. Let $\{e_1, e_2\}$ denote the standard basis in $V = \mathbb{R}^2$, and define $G := \{\pm e_1\}$ and $H := \{e_1, \pm e_2\}$ so that cone (G) and cone (H) are obviously not pointed and form a maximal angle—achieved by the generators themselves—of π. Perturbing the elements of G and H cannot change that maximal angle much, since the perturbed cousins of $-e_1 \in G$ and $e_1 \in H$ will remain almost diametrically opposed.

In contrast, it's easy to see that the maximal angle within a *single* cone is π if and only if the cone is not pointed, making the (dis)continuity of the conic hull in that case moot.

Corollary 6. *Suppose V is a Euclidean space and that $G \in \mathcal{G}(V)$. Then either* cone (G) *is not pointed and $\theta_{max}(K) = \pi$, or the map $X \mapsto \theta_{max}(\text{cone}(X))$ from $\mathcal{G}(V)$ to $[0, \pi)$ is continuous at G.*

4 Conclusions

This is a fairly happy ending. We saw only finite sets of generators, but share that myopia with any algorithm whose input comprises the generators of a cone. That makes Theorem 3 a satisfying computational result quite generally. Our characterization of continuity in the single-cone maximal angle problem is complete. In the two-cone problem we obtained only sufficiency, but that is a two-cone-problem problem: improvements to Corollary 5 will likely require further insights into the behavior of the maximal angle map.

Still, something nags. Luc and Wets were able to establish results for essentially arbitrary sets in Theorem 2, and we would feel better if Theorem 3 was valid for infinite sets. Suppose we ask that our generating sets be merely compact as opposed to finite. Our own results are resilient to such a change, as are Lemmas 2 and 4 of Davis, Wets, and Witzgall. The problem one encounters is that compact sets can generate cones that aren't closed. At the expense of its metric nature, the Pompeiu-Hausdorff distance can be defined with supremums instead of maximums to accommodate them. Will we find salvation if, instead of epsilon-delta, we are willing to settle for a topological notion of continuity? We leave these speculations as such, and the general problem open.

A   What's in a domain?

Oliver Heaviside says [4] that definitions "make themselves, when the nature of the subject has developed itself," and ours are no exception. The candidates discarded would not smell as sweet.

Starting from maximal angles [8], we first tried to adopt the space of all closed convex cones $K \subseteq V$ such that $K \notin \{\{0\}, V\}$ as the codomain of the conic hull. With it came the spherical metric that reinforces the need for K to be nontrivial: the set $K = \{0\}$ doesn't intersect the unit sphere, and while it's possible to define a Pompeiu-Hausdorff distance that works on the empty set, the resulting notion of distance is not metric. So we kept the condition that $K \neq \{0\}$ in furtherance of our stated goal, metric epsilon-delta continuity. The condition that $K \neq V$ was more problematic, and we return to it later.

The decision to restrict the codomain of the conic hull to closed convex cones $K \neq \{0\}$ had immediate repercussions for its domain. Convexity of $\mathrm{cone}\,(G)$ is given, but the most straightforward way to ensure that $\mathrm{cone}\,(G)$ is closed was to make G finite. Moreover the empty set could not belong to the domain because $\mathrm{cone}\,(\emptyset) = \{0\}$ is not admissible. With the sets in our domain nonempty and finite, Pompeiu-Hausdorff was then an obvious choice for a distance because it constitutes a metric on the space of nonempty closed and bounded subsets of the ambient space [1].

We were first inspired to exclude the origin from our generating sets after noticing that $\mathrm{cone}\,(G \cup \{0\}) = \mathrm{cone}\,(G)$ regardless of G; the origin is always redundant. But this turned out to be crucial because the origin can be perturbed to generate an arbitrary ray. It therefore impedes continuity wherever it arises. Retroactively, we observe that the origin belongs to the lineality space of every cone, and the moral of Proposition 4 is that discontinuity comes from having generators in your lineality space. If $0 \in G$, then the argument for Proposition 4 proceeds without Corollary 4, even if $\mathrm{cone}\,(G)$ is pointed. In other words, the presence of the origin in G will lead to discontinuity in all but corner cases. This one unassuming detail is likely to blame for much of our success.

Which brings us back to the case of $K = V$ in the codomain. Remember that we are concerned with open-ball continuity around a given generating set G. The larger problem we were up against was therefore to ensure that all

reasonable notions of a perturbation were captured by open balls in the domain. After some meditation, we were convinced that excluding the origin from the generating sets did not violate that principle, nor did excluding empty or infinite sets *as* generating sets. The same could not be said about sets that generate V, however. The half-space-generating $G \coloneqq \{e_1, \pm e_2\} \subseteq \mathbb{R}^2$, for example, is easily perturbed to obtain a $\tilde{G}$ that generates all of $\mathbb{R}^2$. Disallowing $K = V$ in the codomain would invalidate that perturbation by removing $\tilde{G}$ from the neighborhoods about G. As we were compelled to account for those sorts of perturbations, we were forced to allow $K = V$ in the codomain.

It's only fair at this point to mention a flaw in our own approach. If $G = \{g_1\} \in \mathcal{G}(V)$, then for example $g_2 \coloneqq 2g_1$ constitutes a superfluous generator that is distinct from g_1. Thus upon defining $\tilde{G} \coloneqq \{g_1, g_2\}$ we have $\operatorname{cone}(G) = \operatorname{cone}(\tilde{G})$ despite $\operatorname{haus}(G, \tilde{G}) \neq 0$. For maximum efficacy, it would be preferable if G and $\tilde{G}$ were coincident, but it's not immediately clear how to achieve that. Apropos of nothing else, either g_1 or g_2 can be considered redundant in $\tilde{G}$, and the distinction begins to matter when you want to delete one.

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