Jordan automorphisms and derivatives of symmetric cones

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Abstract

Hyperbolicity cones, and in particular symmetric cones, are of great interest in optimization. Renegar showed that every hyperbolicity cone has a family of derivative cones that approximate it. Ito and Lourenço found the automorphisms of those derivatives when the original cone is generated by rank-one elements, as symmetric cones happen to be. We show that the derivative automorphisms of a symmetric cone are closely related to the automorphisms of its associated Euclidean Jordan algebra. In the process, we find the automorphism group of the quaternion positive-semidefinite cone and list explicitly the Jordan-automorphisms of the quaternion Hermitian matrices. We also address the path-connectedness of the simple Euclidean Jordan-automorphism groups.

Keywords: symmetric cone, rank-one-generated, hyperbolicity cone, derivative relaxation, Euclidean Jordan algebra

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1 Introduction

There are many species of cone in the conic optimization ecosystem, and, true to the metaphor, its occupants fit poorly into a hierarchy. Narrowing our focus around symmetric cones, we do however begin to see something that looks like a family tree. Linear programming was the first conic optimization problem and takes place in the nonnegative orthant $\mathbb{R}^n_+$. Under the usual inner product, the nonnegative orthant forms a polyhedral, self-dual, closed convex cone. Karmarkar showed that linear programs can be solved in polynomial time using interior-point methods with barrier functions [19], and those techniques were soon extended to the second-order cone $\mathcal{L}^n_+$ and the real Hermitian positive-semidefinite cone $\mathcal{H}^n_+(\mathbb{R})$. Those cones are also self-dual (under appropriate inner products), closed, and convex—but no longer polyhedral. When Nesterov and Nemirovskii introduced self-concordant barrier functions [24], they showed that said barriers exist for the nonnegative orthant, second-order cone, and the semidefinite cone. In fact, they proved the existence of a "universal" self-concordant barrier function on any open convex set (think: the interior of your cone), but the argument is nonconstructive.

A *homogeneous* cone is a convex cone whose automorphism group acts transitively on its interior. Following Nesterov and Nemirovskii, Güler showed that the universal self-concordant barrier function is known for (the interior of) every homogeneous cone [13]. Güler also notes that the theory of Jordan algebras can be used to classify self-dual homogeneous cones. Around the same time, Nesterov and Todd were working on efficient methods for "self-scaled" cones [25]. Self-scaled cones turned out to be nothing more than self-dual homogeneous cones, which, in turn, are cones of squares in a Euclidean Jordan algebra [5]. Nowadays they are called *symmetric cones*, and the nonnegative orthant, second-order cone, and semidefinite cone are the most famous examples.

Euclidean Jordan algebras come equipped with a notion of eigenvalues, and every symmetric cone consists of the elements in some Euclidean Jordan algebra whose eigenvalues are nonnegative. These eigenvalues are the roots of a characteristic polynomial, and all such characteristic polynomials have the same degree called the *rank* of the algebra. As a result, the rank of a Euclidean Jordan algebra is the number of eigenvalues (with repetition) that its elements have. Call the product of an element's eigenvalues its determinant. A spectral theorem for Euclidean Jordan algebras then shows that the determinant is obtained from a homogeneous polynomial whose degree is equal to the rank of the algebra. This is all quite analogous to the matrix case; the eigenvalues of an element x in a Euclidean Jordan algebra V are the (necessarily real) roots of the map $\lambda \mapsto \det(\lambda 1_V - x)$.

A homogeneous polynomial $p \in \mathbb{R}[X_1, X_2, \ldots, X_n]$ is *hyperbolic along* $e \in \mathbb{R}^n$ if $p(e) > 0$ and if the roots of $\lambda \mapsto p(\lambda e - x)$ are real for all $x \in \mathbb{R}^n$. Surprisingly, this definition arose not as a generalization of the determinant, but as existence and uniqueness criteria for certain partial differential equations [6, 14]. Nevertheless, we now call the roots of $\lambda \mapsto p(\lambda e - x)$ the eigenvalues of x. The set of elements with only nonnegative eigenvalues,

$$K_{p,e} \coloneqq \{x \in \mathbb{R}^n \mid p(\lambda e - x) \neq 0 \text{ for all } \lambda < 0\},$$

forms a closed convex cone called a *hyperbolicity cone* [6]. The nonnegative orthant, second-order cone, and semidefinite cones were early examples [7].

It was again Güler who noticed that $x \mapsto -\log(p(x))$ is a self-concordant barrier function for a hyperbolicity cone [14], making them vulnerable to the methods of Nesterov and Nemirovskii. Güler also showed that every homogeneous cone is a hyperbolicity cone, giving us the hierarchy,

$$\left\{\mathbb{R}^n_+, \mathcal{L}^n_+, \mathcal{H}^n_+(\mathbb{R})\right\} \subseteq \text{ symmetric } \subseteq \text{ homogeneous } \subseteq \text{ hyperbolicity.}$$

Readers familiar with semidefinite programming will recall that the positive-semidefinite cone is generated by the set of its elements having exactly one nonzero eigenvalue. The same is true of the cone of squares in a Euclidean Jordan algebra, but not generally of a hyperbolicity cone. This encouraged

Ito and Lourenço to define a *rank-one-generated* hyperbolicity cone to be a pointed cone whose extreme rays are generated by elements having only one nonzero eigenvalue [16]. All symmetric cones are rank-one-generated, but we do not know if all (pointed) homogeneous cones are. This puts our hierarchy in peril: do rank-one-generated hyperbolicity cones fit above, below, or beside homogeneous cones?

Ito and Lourenço made significant progress with rank-one-generated hyperbolicity cones. In particular they were able to find the automorphisms of the Renegar derivative cones $K_{p,e}^{(i)}$. Under technical conditions that we omit for now,

$$\operatorname{Aut}\left(K_{p,e}^{(i)}\right) = \operatorname{Aut}\left(K_{p,e}\right) \cap \operatorname{Aut}\left(\mathbb{R}_{+}e\right). \tag{1}$$

The expression on the right, thinly disguised, is familiar in a Euclidean Jordan algebra and is closely related to its Jordan-automorphism group. Without further ado, we state our goal: to specialize Equation (1) to a symmetric cone, connecting the automorphism group of its derivatives to the Jordan automorphism group of the associated Euclidean Jordan algebra. Along the way we'll learn a few things about the Jordan-automorphism groups of simple Euclidean Jordan algebras, and about the automorphism groups of their symmetric cones.

2 Background

2.1 Euclidean Hurwitz algebras

A *Euclidean Hurwitz algebra* is a real algebra $\mathbb{A}$ having a multiplicative unit $1_{\mathbb{A}}$ and an inner product $\langle\cdot,\cdot\rangle : \mathbb{A}^2 \to \mathbb{R}$ such that the associated norm $\|x\| \coloneqq \langle x, x\rangle^{1/2}$ satisfies $\|xy\| = \|x\|\,\|y\|$ for all $x, y \in \mathbb{A}$. This is stronger than the submultiplicativity that you might expect from (say) a matrix norm. Hurwitz's theorem (Faraut and Korányi [5], Theorem V.1.5) tells us that the only Euclidean Hurwitz algebras are the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, and the octonions $\mathbb{O}$.

On any of these algebras an involution $\overline{x} \coloneqq 2\langle x, 1_{\mathbb{A}}\rangle 1_{\mathbb{A}} - x$ called *conjugation* is defined, and we see that $\overline{x}/\|x\|^2$ serves as an inverse to x. In general, conjugation is an anti-homomorphism: $\overline{xy} = \overline{y}\,\overline{x}$ for all $x, y \in \mathbb{A}$. We will always think of $(\mathbb{C}, \mathbb{H}, \mathbb{O})$ as $(2, 4, 8)$-dimensional algebras over $\mathbb{R}$. Each comes with a somewhat standard multiplication table [2] defined on an orthonormal basis $\{e_0, e_1, \ldots, e_N\}$, where $N \in \{1, 3, 7\}$ and e_0 is identified with $1_{\mathbb{R}}$. Conjugation leaves e_0 fixed, but $\overline{e_i} = -e_i$ on the others.

We will deal often with matrices whose entries live in $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$. The space of all $n \times n$ matrices with entries in $\mathbb{A}$ is $\mathbb{A}^{n\times n}$, and in any of these, the identity matrix is I. We define the entrywise conjugate of $X \in \mathbb{A}^{n\times n}$ to be $\overline{X}$, and say that X is *Hermitian* if $X = \overline{X}^T$. The set of all $n \times n$ Hermitian matrices with entries in $\mathbb{A}$ is denoted $\mathcal{H}^n(\mathbb{A})$. Each $X \in \mathbb{A}^{n\times n}$ acts on $\mathbb{A}^n$ by left-multiplication, meaning that we identify $\mathbb{A}^n$ with $\mathbb{A}^{n\times 1}$. When there's an inner-product on $\mathbb{A}^n$, we write X^* for the adjoint of X with respect to it.

The real and complex numbers should be familiar and we assume that you know how they work. The octonions we largely avoid: Baez [2] and Yokota [33] provide background, and we cite important results, but introduce nothing new. That leaves only the quaternions to worry about.

2.1.1 Quaternions

The quaternions are associative but not commutative. This makes linear algebra awkward because it transports us from the realm of vector spaces into that of left- or right-modules. Notably, if we want to do matrix multiplication on the left, then we have to think of "vectors" as being scaled on the right. To scale them on the left, we would need to do matrix multiplication on the right. Our two main sources for the quaternions are Rodman [28] and Tapp [31] who, naturally, disagree on this convention.

Forced to choose, we follow Rodman. Thus $\mathbb{H}^n$ is a right module over $\mathbb{H}$, with an inner product (Rodman, Definition 3.1.2) defined by $\langle x, y \rangle \coloneqq \sum_{i=1}^{n} \overline{y_i} x_i$ for $x, y \in \mathbb{H}^n$. Neither author explicitly says so, but $\overline{X}^T$ serves as the adjoint of $X \in \mathbb{H}^{n \times n}$ with respect to this inner product. We are therefore allowed to write X^* for the conjugate-transpose of a quaternion matrix, as we will do for real and complex matrices.

The invertible elements in $\mathbb{H}^{n \times n}$ form a group $\mathrm{GL}_n(\mathbb{H})$, and the isometries on $\mathbb{H}^n$ form a subgroup of $\mathrm{GL}_n(\mathbb{H})$ called the *symplectic group* (Tapp, Definition 3.8), whose members equivalently

1. preserve the inner product (by definition);
2. preserve the norm induced by the inner product (Tapp, Proposition 3.11);
3. have adjoints equal to their inverses (Tapp, Proposition 3.9).

Rodman's inner product differs from the one used by Tapp, but they are conjugate to one another. As a result, the two authors' notions of adjoint and isometry are the same. To abate the proliferation of notation, we take advantage of the third equivalent condition above to make the following definition.

Definition 1. If $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, then

$$\mathrm{Isom}(\mathbb{A}^n) \coloneqq \{U \in \mathrm{GL}_n(\mathbb{A}) \mid U^* U = I\}$$

denotes the set of real orthogonal, complex unitary, or quaternion symplectic n-by-n matrices.

2.2 Euclidean Jordan algebras

Faraut and Korányi is our primary source for Euclidean Jordan algebras [5]. It's the standard reference in optimization, and our results are more interesting in that context. We will recall many important definitions and theorems, but shall out of necessity assume that the reader is acquainted with the first five chapters of Faraut and Korányi. For instance, all Jordan algebras are power-associative, allowing us to write x^2 unambiguously in what follows.

Definition 2. A *Jordan algebra* is a commutative algebra $(V, \circ)$ over a field of characteristic not equal to two whose product satisfies the identity $x \circ \left(x^2 \circ y\right) = x^2 \circ (x \circ y)$ for all $x, y \in V$. A *Euclidean Jordan algebra* is a finite-dimensional real unital Jordan algebra in which the only solution to $x^2 + y^2 = 0$ is $x = y = 0$. A *simple* (Euclidean) Jordan algebra is one that has no nontrivial ideals. The *cone of squares* in V is the set $\left\{x^2 \mid x \in V\right\}$. If $x \in V$, then the degree of x is the dimension of the subalgebra generated by x, and the *rank* of V is the maximal degree of its elements.

Our *Euclidean* is more commonly called *formally-real*. However, modulo a distinguished "associative" inner product, Euclidean and formally-real Jordan algebras are equivalent (Faraut and Korányi [5], Section III.1 and Proposition VIII.4.2). This means that the inner product typically associated with a Euclidean Jordan algebra is less important than one might expect. Case in point: this paper is about Jordan isomorphisms, and Jordan isomorphisms do not have to respect inner products. Definition 2 lets us omit them.

Definition 3. If $(V, \circ)$ and $(W, \bullet)$ are two Jordan algebras, then $\varphi : V \to W$ is a *Jordan isomorphism* between V and W if it is linear, invertible, and if $\varphi(x \circ y) = \varphi(x) \bullet \varphi(y)$ for all $x, y \in V$. Two Jordan algebras are *Jordan-isomorphic* if there exists a Jordan isomorphism between them. A *Jordan automorphism* is a Jordan isomorphism from a Jordan algebra to itself. We write $\mathrm{JAut}(V)$ for the Jordan automorphism group of V, omitting the multiplication, and promise that no ambiguity will arise.

Theorem 1 (Faraut and Korányi [5], Proposition III.4.4 and Chapter V)**.** *Every Euclidean Jordan algebra is the orthogonal direct sum of a unique set of nontrivial simple algebras, and every nontrivial simple algebra is Jordan-isomorphic to a member of one of the five families,*

1. *the Jordan spin algebras $\mathcal{L}^n$, for $n \geq 1$;*
2. *the algebras of $n \times n$ real Hermitian matrices $\mathcal{H}^n(\mathbb{R})$, for $n \geq 3$;*
3. *the algebras of $n \times n$ complex Hermitian matrices $\mathcal{H}^n(\mathbb{C})$, for $n \geq 3$;*
4. *the algebras of $n \times n$ quaternion Hermitian matrices $\mathcal{H}^n(\mathbb{H})$, for $n \geq 3$;*
5. *the Albert algebra of 3×3 octonion Hermitian matrices $\mathcal{H}^3(\mathbb{O})$.*

The restrictions on n in Theorem 1 ensure that, up to Jordan-isomorphism, no algebra appears twice in the list. If we instead group them "up to notation," the list gets even shorter. The Hermitian matrix algebras $\mathcal{H}^n(\mathbb{A})$ for $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ all have Jordan product $X \circ Y \coloneqq (XY + YX)/2$ with the identity matrix serving as the unit element. The Jordan spin algebras $\mathcal{L}^n$ live in $\mathbb{R} \times \mathbb{R}^{n-1}$ with unit element $\left(1, \tilde{0}\right)$ and Jordan product

$$x \circ y = \begin{bmatrix} x_0 \\ \tilde{x} \end{bmatrix} \circ \begin{bmatrix} y_0 \\ \tilde{y} \end{bmatrix} \coloneqq \begin{bmatrix} x_0 y_0 + \langle \tilde{x}, \tilde{y} \rangle \\ y_0 \tilde{x} + x_0 \tilde{y} \end{bmatrix}.$$

Definition 4. Suppose K is a convex cone in some finite-dimensional real inner-product space V. The automorphism group of K is,

$$\operatorname{Aut}(K) \coloneqq \{\varphi : V \to V \mid \varphi \text{ is linear, invertible, and } \varphi(K) = K\}.$$

If $e \in V$, then the corresponding stabilizer subgroup is,

$$\operatorname{Aut}(K)_e = \{\varphi \in \operatorname{Aut}(K) \mid \varphi(e) = e\}.$$

If $K \cap -K = \{0\}$, then K is *pointed*. If K is self-dual, and if $\operatorname{Aut}(K)$ acts transitively on the interior of K, then K is *symmetric*. We write $\operatorname{int}(K)$ for the interior of K.

Remark 1. With this definition, the cone of squares in a Euclidean Jordan algebra is symmetric, but symmetric cones are often defined as the *interior* of what we have called a symmetric cone. For optimization, a closed set is nice; for differential geometry, not so much. There are generally many symmetric cones in each Euclidean Jordan algebra, but only one that we want to talk about. Having defined a symmetric cone to be closed, that cone is the cone of squares. The interior/closure definitions are easy to switch between in any case, and the choice does not affect the automorphism groups of the cones.

Each element in a Euclidean Jordan algebra has a spectral decomposition that should look familiar from linear algebra. The number of terms in these decompositions is the rank of the algebra (Definition 2) and does not depend on the element [5], it is akin to the size n of an n-by-n Hermitian matrix.

Theorem 2 (Faraut and Korányi [5], Theorems III.1.1–2)**.** *If $(V, \circ)$ is a Euclidean Jordan algebra of rank r and if $x \in V$, then there exists a set of idempotents $\{c_1, c_2, \ldots, c_r\}$ in V that sum to 1_V and real numbers $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_r(x)$ such that $c_i \circ c_j = 0$ when $i \neq j$, and*

$$x = \lambda_1(x)\, c_1 + \lambda_2(x)\, c_2 + \cdots + \lambda_r(x)\, c_r.$$

This is the *spectral decomposition* of x, and the $\lambda_i(x)$ are the *eigenvalues* of x. We write $\operatorname{rank}_V(x)$ to denote the number of nonzero eigenvalues possessed by $x \in V$, with the Jordan product understood from context. The repurposing of the word "rank" is not great, but the quantity $\operatorname{rank}_V(x)$ generalizes the usual rank of a Hermitian matrix x in $V = \mathcal{H}^n(\mathbb{R})$ or $V = \mathcal{H}^n(\mathbb{C})$. The cone of squares in a Euclidean Jordan algebra likewise generalizes the positive-semidefinite cones, in that its elements are those having only nonnegative eigenvalues. This is apparent from the identity $\lambda_i\left(x^2\right) = \lambda_i(x)^2$ that can be deduced from the spectral decomposition.

Corollary 1. *The cone of squares in a Euclidean Jordan algebra is the set of elements having only nonnegative eigenvalues.*

The eigenvalues in the spectral decomposition are the roots of a univariate characteristic polynomial, and each characteristic polynomial arises from a multivariate polynomial on the ambient algebra. The following is the author's edition [26] of Faraut and Korányi's Proposition II.2.1 that explicitly mentions the basis $\mathbf{b}$. From now on, we write $\mathbf{b}(x)$ for the $\mathbf{b}$-coordinates of x.

Theorem 3. *If $(V, \circ)$ is a Euclidean Jordan algebra of rank r and dimension n with basis $\mathbf{b}$, then there exist $a_0, a_1, \ldots, a_{r-1} \in \mathbb{R}[X_1, X_2, \ldots, X_n]$ such that the characteristic polynomial of any $x \in V$ is*

$$\Lambda^r + \sum_{i=0}^{r-1} a_i(\mathbf{b}(x))\,\Lambda^i \in \mathbb{R}[\Lambda].$$

The coefficient polynomials a_i are homogeneous of degree $r-i$, and the eigenvalues of any $x \in V$ are the (necessarily real) roots of its characteristic polynomial.

More analogies can be drawn from this. For example,

Definition 5. In the setting of Theorem 3, we define a determinant on V with respect to the basis $\mathbf{b}$ by $\det_{\mathbf{b}} \coloneqq (-1)^r a_0$.

From Theorem 3 it follows that $\det_{\mathbf{b}}$ is homogeneous of degree r, the rank of the algebra. As you would hope, the determinant of (the $\mathbf{b}$-coordinates of) an element is the product of its eigenvalues. From the spectral decomposition it follows that the r real roots of $\lambda \mapsto \det_{\mathbf{b}}(\mathbf{b}(\lambda 1_V - x))$ are the eigenvalues of $x \in V$. While it is possible to make this definition basis-agnostic, it will be useful in a moment to have a polynomial defined on $\mathbb{R}^n$.

2.3 Hyperbolic polynomials

Preferring narrative continuity over historical accuracy, we tailor our definitions of hyperbolic polynomials and hyperbolicity cones to fit the previous section.

Definition 6. A polynomial $p \in \mathbb{R}[X_1, X_2, \ldots, X_n]$ is *hyperbolic along* $e \in \mathbb{R}^n$ if it is homogeneous, if $p(e) > 0$, and if for all $x \in \mathbb{R}^n$ the map $\lambda \mapsto p(\lambda e - x)$ has only real roots. In that case $\deg(p)$ denotes the degree of p, the univariate polynomial defined by $\lambda \mapsto p(\lambda e - x)$ is the characteristic polynomial of x, and the real roots of that characteristic polynomial are the *eigenvalues* of x—all with respect to p and e. The set

$$K_{p,e} \coloneqq \{x \in \mathbb{R}^n \mid p(\lambda e - x) \neq 0 \text{ for all } \lambda < 0\}$$

is the *hyperbolicity cone* of p along e.

All hyperbolicity cones are closed convex cones[1]. This was originally proved by Lars Gårding [6], but updated arguments have been given by others including Güler [14] and Renegar [27]. It is interesting to note that if $\tilde{e} \in \operatorname{int}(K_{p,e})$, then $K_{p,\tilde{e}} = K_{p,e}$ is the same cone [14, 27]. As a result, it is desirable for certain properties to be independent of the point $\tilde{e}$. One such is the rank of an element [16]. Given $x \in \mathbb{R}^n$, the eigenvalues $\lambda_i(x)$ with respect to p and e will typically depend on both p and e, but the number of nonzero eigenvalues remains the same with respect to any other $\tilde{e} \in \operatorname{int}(K_{p,e})$. Renegar proved this by counting the multiplicity of zero as an eigenvalue [27].

[1] Remark 1 is relevant here too.

Definition 7. If $p \in \mathbb{R}[X_1, X_2, \ldots, X_n]$ is hyperbolic along e, then $\operatorname{rank}_{p,e}(x)$ is the *rank* of $x \in \mathbb{R}^n$ with respect to p and e, the number of nonzero eigenvalues x has with respect to p and e.

Suppose V is a Euclidean Jordan algebra of rank r. Recall from Theorem 2 that every $x \in V$ has a set of real eigenvalues $\lambda_1(x), \lambda_2(x), \ldots, \lambda_r(x)$. If we choose a basis $\mathbf{b}$ for V, then using Theorem 3, it is straightforward to define p and e such that p is hyperbolic along e and the $\lambda_i(x)$ are the real roots of the map $\lambda \mapsto p(\lambda e - \mathbf{b}(x))$. In other words, we can use the basis representation to make the two notions of "eigenvalue" agree, after which Corollary 1 shows that the hyperbolicity cone in $\mathbb{R}^n$ is the coordinate representation of the cone of squares in V.

Theorem 4 (Ito and Lourenço [16], Proposition 3.8)**.** *If $(V, \circ)$ is a Euclidean Jordan algebra of rank r with basis $\mathbf{b}$, then $p \coloneqq \det_{\mathbf{b}}$ is a hyperbolic polynomial of degree r along $e \coloneqq \mathbf{b}(1_V)$, and the Jordan-algebraic eigenvalues of any $x \in V$ are the roots of $\lambda \mapsto p(\lambda e - \mathbf{b}(x))$. It follows that $K_{p,e} = \mathbf{b}\left(\left\{x^2 \;\middle|\; x \in V\right\}\right)$ and that $\operatorname{rank}_{p,e}(\mathbf{b}(x)) = \operatorname{rank}_V(x)$ for all $x \in V$.*

The cone of squares in a Euclidean Jordan algebra has another important property: its extreme directions are generated by rank-one elements. (In view of Theorem 4, either interpretation of "rank" is valid.) Ito and Lourenço proved this, and it can be inferred from Proposition IV.3.2 in Faraut and Korányi [5]. This leads Ito and Lourenço to define *rank-one-generated* hyperbolicity cones, a family that lies somewhere between symmetric cones and hyperbolicity cones.

Definition 8. A hyperbolicity cone $K_{p,e}$ is *rank-one-generated* if it is pointed and if its extreme rays are all of the form $\mathbb{R}_+ x$ for some x having $\operatorname{rank}_{p,e}(x) = 1$.

In general, whether or not $K_{p,e}$ is rank-one-generated depends on the polynomial p. Two polynomials p and $\tilde{p}$ can have $K_{p,e} = K_{\tilde{p},e}$ despite only one of them being rank-one-generated (Ito and Lourenço [16], Remark 3.2). We however are interested in symmetric cones, all of which we now know are rank-one-generated with respect to certain determinant polynomials—the only polynomials we use.

One last concept is needed. Given a polynomial p hyperbolic along e, Renegar [27] defines a derivative polynomial $p'_{\tilde{e}}$ to be the directional derivative of p along $\tilde{e}$. It is then not hard to see that $p'_{\tilde{e}}$ is hyperbolic along $\tilde{e}$ if $\tilde{e} \in \operatorname{int}(K_{p,e})$, and of course there is an associated derivative cone $K_{p'_{\tilde{e}},\tilde{e}}$. Different choices of $\tilde{e}$ produce different $p'_{\tilde{e}}$, which in turn produce different derivative cones. Many nice results hold only when the derivative is taken in the original direction of hyperbolicity, e. Ito and Lourenço work under this assumption [16], so we bake it into the definition.

Definition 9. If $p \in \mathbb{R}[X_1, X_2, \ldots, X_n]$ is hyperbolic along e, then its ith *derivative polynomial* along e is the directional derivative $p_e^{(i)} \coloneqq D_e^{(i)}(p)$, itself hyperbolic along e. The associated *derivative cone* $K_{p_e^{(i)},e}$ is abbreviated $K_{p,e}^{(i)}$.

In many cases, Ito and Lourenço were able to find the automorphisms of these derivative cones. In transcribing their result—which brings us up to date—we have made use of the identity

$$\operatorname{Aut}(K_{p,e}) \cap \operatorname{Aut}(\mathbb{R}_+ e) = \mathbb{R}_{++} \operatorname{Aut}(K_{p,e})_e .$$

Theorem 5 (Ito and Lourenço, Theorem 3.15)**.** *If $K_{p,e}$ is rank-one-generated in $\mathbb{R}^n$ with $n \geq 3$, $\deg(p) \geq 4$, and $i \in \{1, 2, \ldots, \deg(p) - 3\}$, then*

$$\operatorname{Aut}\left(K_{p,e}^{(i)}\right) = \mathbb{R}_{++} \operatorname{Aut}(K_{p,e})_e .$$

3 Decomposing automorphisms groups

If K is a symmetric cone, then it is the cone of squares in some Euclidean Jordan algebra V. Choosing p and e appropriately (as in Theorem 4), we can use Theorem 5 to find the automorphisms of the derivatives of $K = K_{p,e}$, but only if we know what's in the group $\operatorname{Aut}(K)_{1_V}$. The goals of this section are to show that $\operatorname{Aut}(K)_{1_V} = \operatorname{JAut}(V)$, and that $\operatorname{JAut}(V)$ and $\operatorname{Aut}(K)$ both have decompositions in the style of Theorem 1. In subsequent sections, we study the groups $\operatorname{JAut}(V_i)$ and $\operatorname{Aut}(K_i)$ that arise in those decompositions. From the parts we can assemble the derivative automorphisms of the original cone.

The identity $\operatorname{Aut}(K)_{1_V} = \operatorname{JAut}(V)$ goes back to Vinberg who stated it without proof in 1965 [32]. Chua recognized the need for a proof in 2008 and supplied one using the characteristic function of the cone [4]. An equivalent result appears as Theorem 2.80 in Alfsen and Schultz whose argument is pleasantly calculus-free [1]. We give yet another proof, using the decomposition of a Euclidean Jordan algebra into simple components, where Gowda records the following based solely on results in Faraut and Korányi [5].

Lemma 1 (Gowda [8], Theorem 8)**.** *If $(V, \circ)$ is a simple Euclidean Jordan algebra with cone of squares K, then* $\operatorname{JAut}(V) = \operatorname{Aut}(K)_{1_V}$.

The inclusion $\operatorname{JAut}(V) \subseteq \operatorname{Aut}(K)_{1_V}$ is "easy," so we focus on showing that $\operatorname{Aut}(K)_{1_V}$ consists only of Jordan automorphisms. We know from Theorem 1 how a Euclidean Jordan algebra decomposes into simple factors, and we plan to decompose $\operatorname{Aut}(K)_{1_V}$ along the same lines so that Lemma 1 applies componentwise. A germane decomposition of $\operatorname{Aut}(K)$ was found by Horne and is based on a decomposition of K itself that is valid when K is symmetric [17]. It will be expedient to state definitions and results only for symmetric cones.

Definition 10. A symmetric cone is *reducible* if it is the direct sum of two nontrivial symmetric cones, and *irreducible* if not.

In an intuitive way, reducible cones reduce to irreducible ones. For symmetric cones, the proof of this claim lay hidden within Theorem 1.

Proposition 1 (Faraut and Korányi [5], Proposition III.4.5)**.** *If K is a symmetric cone in a Euclidean Jordan Algebra V, then K is the orthogonal direct sum of a unique set of nontrivial irreducible symmetric cones. In particular, the cone of squares in V is the orthogonal direct sum of the cones of squares in its simple factors a la Theorem 1.*

Horne used this decomposition to show that the automorphism group of a "full" cone decomposes into a product of the automorphism groups of its irreducible components [17]. We specialize Horne's result by assuming that the cones are symmetric and that their factors are pairwise either non-isomorphic or equal. Specifically, we exclude the possibility that two factors are isomorphic but not equal. And since Proposition 1 produces an orthogonal decomposition, we opt for a Cartesian product representation to make the orthogonality explicit. Finally, we've added the claim that stabilizer subgroups factor in the same way; this is a new claim, but straightforward to check given Horne's result.

Theorem 6 (Horne [17], Lemma 4.1 and Theorem 4.2)**.** *If $J = \bigtimes_{i=1}^{m} J_i$ is a Cartesian product of nontrivial symmetric cones $J_i = \bigtimes_{j=1}^{m_i} K_i$, and if each K_i is irreducible and symmetric and not isomorphic to K_ℓ for $\ell \neq i$, then*

$$\operatorname{Aut}(J) = \bigtimes_{i=1}^{m} \operatorname{Aut}(J_i)$$

where

$$\operatorname{Aut}(J_i) = \left(\bigtimes_{j=1}^{m_i} \operatorname{Aut}(K_i) \right) \Sigma_{m_i}$$

and where Σ_{m_i} denotes the group of permutations of the m_i factors of J_i. Moreover if $x = (x_1, x_2, \ldots, x_m) \in J$ with $x_i = (\xi_i, \xi_i, \ldots, \xi_i) \in J_i$, then

$$\operatorname{Aut}(J)_x = \bigtimes_{i=1}^{m} \operatorname{Aut}(J_i)_{x_i}$$

where

$$\operatorname{Aut}(J_i)_{x_i} = \left(\bigtimes_{j=1}^{m_i} \operatorname{Aut}(K_i)_{\xi_i} \right) \Sigma_{m_i}.$$

To apply this decomposition, we will start with Theorem 1, and use a Jordan isomorphism to turn the orthogonal direct sum into a Cartesian product. Later, we use the following not-quite-trivial result to make the isomorphism go away.

Lemma 2. *Suppose $L : V \to W$ is an invertible linear map between two real vector spaces V and W. If K is a convex cone in V and if $x \in V$, then*

$$L \operatorname{Aut}(K)_x L^{-1} = \operatorname{Aut}(L(K))_{L(x)}.$$

In particular, with $x = 0$, we have $L \operatorname{Aut}(K) L^{-1} = \operatorname{Aut}(L(K))$.

Theorem 7. *If $(V, \circ)$ is a Euclidean Jordan algebra with cone of squares K, then $\operatorname{JAut}(V) = \operatorname{Aut}(K)_{1_V}$.*

Proof. The inclusion $\operatorname{JAut}(V) \subseteq \operatorname{Aut}(K)_{1_V}$ is immediate because Jordan automorphisms preserve multiplication and therefore preserve both the cone of squares and the unit element.

For the other inclusion, our first step is to obtain a cone that satisfies the prerequisites of Theorem 6. According to Theorem 1, V is an orthogonal direct sum of simple algebras, each of which is Jordan-isomorphic to a single canonical representative. The map $\sum_{i=1}^{m} x_i \mapsto (x_1, x_2, \ldots, x_m)$ taking the direct sum to a Cartesian product is a Jordan isomorphism, as is the map sending each factor to its canonical representative. We may therefore suppose (by composing them) that there exists a single Jordan isomorphism $\varphi : V \to W$ such that

$$\varphi(V) = W = \bigtimes_{i=1}^{m} W_i = \bigtimes_{i=1}^{m} \left(\bigtimes_{j=1}^{m_i} V_i \right)$$

with W_i consisting of m_i copies of the simple Euclidean Jordan algebra V_i and where V_i, V_ℓ are not Jordan-isomorphic unless $i = \ell$ (in which case they are equal). It follows that

$$\varphi(K) = J = \bigtimes_{i=1}^{m} J_i = \bigtimes_{i=1}^{m} \left(\bigtimes_{j=1}^{m_i} K_i \right)$$

satisfies the prerequisites for Theorem 6. Moreover $1_W = (1_{W_1}, 1_{W_2}, \ldots, 1_{W_m}) \in J$ with each $1_{W_i} = (1_{V_i}, 1_{V_i}, \ldots, 1_{V_i})$. We may thus apply Theorem 6 and Lemma 1 in succession to decompose $\operatorname{Aut}(J)_{1_W}$ into a Cartesian product that is easily seen to be contained in $\operatorname{JAut}(W)$.

A minor obstacle remains: we have the result for W, but we need it for $V = \varphi^{-1}(W)$. To get around that, first convince yourself that $\operatorname{JAut}(W) = \varphi \operatorname{JAut}(V) \varphi^{-1}$, and then use Lemma 2 to cancel the conjugation,

$$\varphi \operatorname{JAut}(V) \varphi^{-1} = \operatorname{JAut}(W) = \operatorname{Aut}(J)_{1_W} = \varphi \operatorname{Aut}(K)_{1_V} \varphi^{-1}. \qquad \square$$

In this argument we recognize an important decomposition of $\operatorname{JAut}(W)$ first discovered by Gowda and Jeong [9]. A proof using Theorem 7 is almost trivial because the guts are hidden in the prerequisites. Similar decompositions are known for both the identity path-component of the cone's automorphism group (Faraut and Korányi [5], Proposition III.4.5), and for the structure group of the algebra (Koecher [20], Chapter IV, Section 6, Theorem 10).

Theorem 8. *If $W = \bigtimes_{i=1}^{m} W_i$ is a Cartesian product of nontrivial Euclidean Jordan algebras $W_i = \bigtimes_{j=1}^{m_i} V_i$, and if each V_i is simple and not isomorphic to V_ℓ for $\ell \neq i$, then*

$$\operatorname{JAut}(W) = \bigtimes_{i=1}^{m} \operatorname{JAut}(W_i)$$

where

$$\operatorname{JAut}(W_i) = \left(\bigtimes_{j=1}^{m_i} \operatorname{JAut}(V_i) \right) \Sigma_{m_i}$$

and where Σ_{m_i} denotes the group of permutations of the m_i factors of W_i.

Proof. Recall that $1_W = (1_{W_1}, 1_{W_2}, \ldots, 1_{W_m})$ and $1_{W_i} = (1_{V_i}, 1_{V_i}, \ldots, 1_{V_i})$ from the proof of Theorem 7. Take $\mathrm{JAut}(W) = \mathrm{Aut}(J)_{1_W}$ and decompose the latter using Theorem 6. Afterwards, apply Theorem 7 again on each factor. $\square$

There's plenty of low-hanging fruit to be plucked with Theorem 7. Many results based on Lemma 1 effortlessly extend from a simple algebra to the general case. For example, the "polar decomposition" in Faraut and Korányi's Theorem III.5.1 extends to the full automorphism group of any symmetric cone. To explain it, we must quickly define the quadratic representation of our algebra [5].

Definition 11. If $(V, \circ)$ is a Jordan algebra, then we define on V

$$\begin{aligned} L_x &\coloneqq y \mapsto x \circ y, \text{ and} \\ P_x &\coloneqq 2\left(L_x\right)^2 - L_{x^2}, \end{aligned}$$

both of which are linear. The "left multiplication by" map $x \mapsto L_x$ is itself linear. The map $x \mapsto P_x$ is called the *quadratic representation* of V and is not.

Proposition 2 (polar decomposition)**.** *If V is a Euclidean Jordan algebra with cone of squares K, then* $\mathrm{Aut}(K) = P_{\mathrm{int}(K)}\, \mathrm{JAut}(V)$.

It is encouraging to note that Larotonda and Luna, in their Proposition 3.23, have recently extended this result to a more general JB-algebra [21]. We will need the polar decomposition later, but otherwise resist the temptation to tangentially exploit Theorem 7.

4 Symmetric cone automorphisms

To recap: in any Euclidean Jordan algebra V with cone of squares K, we now know that the Jordan-automorphism group is $\mathrm{Aut}(K)_{1_V}$, and it decomposes into a product of subgroups $\mathrm{Aut}(K_i)_{1_{V_i}}$ that correspond to the simple components V_i of V. There are essentially five families from which to draw the V_i, and therefore five families of irreducible symmetric cones K_i that we might encounter. So what are the automorphism groups $\mathrm{Aut}(K_i)$? If we can find them, then, up to Jordan-isomorphism, we will know the Jordan-automorphism group of any Euclidean Jordan algebra. In $\mathcal{L}^n$, $\mathcal{H}^n(\mathbb{R})$, and $\mathcal{H}^n(\mathbb{C})$, these groups are known. We intend to add $\mathcal{H}^n(\mathbb{H})$ to the list. Towards that end, the first question we might ask is, what *is* the cone of squares in $\mathcal{H}^n(\mathbb{H})$?

Proposition 3. *In the Euclidean Jordan algebra $\mathcal{H}^n(\mathbb{H})$, the cone of squares is the quaternion positive-semidefinite cone $\mathcal{H}^n_+(\mathbb{H})$, and*

$$\mathrm{Aut}\left(\mathcal{H}^n_+(\mathbb{H})\right) = \left\{X \mapsto A^* X A \mid A \in \mathrm{GL}_n(\mathbb{H})\right\}.$$

Proof. Let K denote the cone of squares in $\mathcal{H}^n(\mathbb{H})$. If $X \in \mathcal{H}^n_+(\mathbb{H})$, then Theorem 5.3.6 of Rodman says that we can diagonalize $X = UDU^*$ by $U \in \mathrm{Isom}(\mathbb{H}^n)$, and his Proposition 5.3.8 shows that the diagonal entries of D are real and nonnegative. Thus we may define $Y := U\sqrt{D}U^*$, and $X = Y^2$, being a square, is in K.

Conversely, suppose $X = Y^2$ for some $Y \in \mathcal{H}^n(\mathbb{H})$. Using Rodman's Theorem 5.3.6 again, we can diagonalize $Y = UDU^*$ to conclude that $X = UD^2U^*$ for some real diagonal matrix D. Proposition 5.3.7 shows that the (nonnegative) entries of D^2 are the eigenvalues of X, and Proposition 5.3.8 lets us conclude that X is positive-semidefinite.

Finally, Theorem 4.1.10 in Rodman gives us the invertible transformations that preserve the property of having zero negative eigenvalues, which are precisely the automorphisms of $\mathcal{H}^n_+(\mathbb{H})$ in light of Rodman's Proposition 5.3.8. $\square$

We now collect what is known about the automorphism groups of the cones of squares in the five simple families of Theorem 1. Sznajder was the first to find an explicit description of the Lorentz cone automorphisms [30]. Ours looks a bit different, and the proof using the quadratic representation is new, but the two are of course equivalent.

Theorem 9. *If $m, n \in \mathbb{N}$ with $m \geq 1$, then*

$$\operatorname{Aut}\left(\mathcal{L}^m_+\right) = \left\{ \begin{bmatrix} x_0^2 + \|\tilde{x}\|^2 & 2x_0\tilde{x}^T U \\ 2x_0\tilde{x} & 2\tilde{x}\tilde{x}^T U + \left(x_0^2 - \|\tilde{x}\|^2\right) U \end{bmatrix} \;\middle|\; \begin{array}{l} x_0 \in \mathbb{R}, \tilde{x} \in \mathbb{R}^{m-1} \\ 0 \leq \|\tilde{x}\| < x_0 \\ U \in \operatorname{Isom}\left(\mathbb{R}^{m-1}\right) \end{array} \right\},$$

and

$$\begin{aligned} \operatorname{Aut}\left(\mathcal{H}^n_+(\mathbb{R})\right) &= \left\{X \mapsto U^*XU \mid U \in \mathrm{GL}_n(\mathbb{R})\right\}, \\ \operatorname{Aut}\left(\mathcal{H}^n_+(\mathbb{C})\right) &= \left\{X \mapsto U^*XU \mid U \in \mathrm{GL}_n(\mathbb{C})\right\} \\ &\quad \cup \left\{X \mapsto U^*\overline{X}U \;\middle|\; U \in \mathrm{GL}_n(\mathbb{C})\right\}, \\ \operatorname{Aut}\left(\mathcal{H}^n_+(\mathbb{H})\right) &= \left\{X \mapsto U^*XU \mid U \in \mathrm{GL}_n(\mathbb{H})\right\}. \end{aligned}$$

Proof. The characterizations of $\operatorname{Aut}\left(\mathcal{H}^n_+(\mathbb{R})\right)$, $\operatorname{Aut}\left(\mathcal{H}^n_+(\mathbb{C})\right)$, and $\operatorname{Aut}\left(\mathcal{H}^n_+(\mathbb{H})\right)$ all follow from inertia theorems, with the quaternion case being Proposition 3. For $\mathbb{R}$ and $\mathbb{C}$, use Theorem 2 of Schneider [29].

Based on the work of Loewy and Schneider [22], Gowda, Sznajder, and Tao [11] showed in their Example 2.1 that

$$\operatorname{JAut}\left(\mathcal{L}^m\right) = \left\{\mathrm{id}_{\mathbb{R}} \times U \;\middle|\; U \in \operatorname{Isom}\left(\mathbb{R}^{m-1}\right)\right\}.$$

We want to substitute this into the polar decomposition of $\operatorname{Aut}\left(\mathcal{L}^m_+\right)$ from Proposition 2, but to do so, we need the form of an arbitrary P_x where $x \in \operatorname{int}\left(\mathcal{L}^m_+\right)$. Fortunately, it's easy to see by checking its action on the standard basis that if we write $x = (x_0, \tilde{x}) \in \mathbb{R} \times \mathbb{R}^{m-1}$ in block form, then

$$L_x = \begin{bmatrix} x_0 & \tilde{x}^T \\ \tilde{x} & x_0 I \end{bmatrix}.$$

With $P_x = 2\left(L_x\right)^2 - L_{x^2}$, a boring computation suffices to find the blocks of the matrix. The conditions on x_0 and $\tilde{x}$ come from $x = (x_0, \tilde{x}) \in \operatorname{int}\left(\mathcal{L}^m_+\right)$. $\square$

5 Simple EJA automorphisms

With $\operatorname{Aut}(K)$ mostly in hand, we can put Theorem 7 to work.

Theorem 10. *If $m, n \in \mathbb{N}$ with $m \geq 1$ and $n \geq 3$, then*

$$\begin{aligned}
\operatorname{JAut}(\mathcal{L}^m) &= \left\{ \mathrm{id}_{\mathbb{R}} \times U \;\middle|\; U \in \operatorname{Isom}\left(\mathbb{R}^{m-1}\right) \right\}, \\
\operatorname{JAut}(\mathcal{H}^n(\mathbb{R})) &= \left\{ X \mapsto U^* X U \mid U \in \operatorname{Isom}(\mathbb{R}^n) \right\}, \\
\operatorname{JAut}(\mathcal{H}^n(\mathbb{C})) &= \left\{ X \mapsto U^* X U \mid U \in \operatorname{Isom}(\mathbb{C}^n) \right\} \\
&\quad \cup \left\{ X \mapsto U^* \overline{X} U \;\middle|\; U \in \operatorname{Isom}(\mathbb{C}^n) \right\}, \\
\operatorname{JAut}(\mathcal{H}^n(\mathbb{H})) &= \left\{ X \mapsto U^* X U \mid U \in \operatorname{Isom}(\mathbb{H}^n) \right\}, \\
\operatorname{JAut}\left(\mathcal{H}^3(\mathbb{O})\right) &= \textit{the exceptional Lie group } F_4.
\end{aligned}$$

Proof. The first four follow from Theorems 7 and 9 and the stipulation that Jordan automorphisms preserve the unit. Chevalley and Schafer proved that F_4 is the identity path-component of $\operatorname{JAut}\left(\mathcal{H}^3(\mathbb{O})\right)$, which is path-connected, making the identity path-component the whole thing [3, 33]. □

Remark 2. Theorem 6.5 of Huang characterizes the Jordan-automorphism group of the $n \times n$ Hermitian matrices when $n \geq 3$ and when the entries come from a division ring that has an involution [15]. In particular it applies over $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$, but not $\mathbb{O}$. In an alternate universe we could start by listing the automorphisms of $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ using Proposition 2.4.7 of Rodman [28] for $\mathbb{H}$, and then deduce $\operatorname{JAut}(\mathcal{H}^n(\mathbb{R}))$, $\operatorname{JAut}(\mathcal{H}^n(\mathbb{C}))$, and $\operatorname{JAut}(\mathcal{H}^n(\mathbb{H}))$ from Huang.

Remark 3. Using the embedding $\mathbb{R} \hookrightarrow \mathbb{R}I$, it's also possible to squeeze isomorphic representations of $\operatorname{JAut}(\mathcal{H}^n(\mathbb{R}))$ and $\operatorname{JAut}(\mathcal{H}^n(\mathbb{H}))$ out of Theorem 6 in Kalisch [18]. The same cannot be said of $\operatorname{JAut}(\mathcal{H}^n(\mathbb{C}))$, however, because complex conjugation is not a complex-linear involution.

The Jordan-automorphism groups of $\mathcal{L}^n$ and $\mathcal{H}^n(\mathbb{R})$ were already known to Gowda, Tao, and Sznajder [11]. A claim was made for $\operatorname{JAut}(\mathcal{H}^n(\mathbb{C}))$, but with the family of transformations involving conjugation omitted [12, 10]. In a footnote, Vinberg warns us not to overlook these, suggesting that $\operatorname{JAut}(\mathcal{H}^n(\mathbb{R}))$ and $\operatorname{JAut}(\mathcal{H}^n(\mathbb{H}))$ were known to him as well [32]. It is worth confirming that the extra family in $\operatorname{JAut}(\mathcal{H}^n(\mathbb{C}))$ is not redundant.

Example 1. Suppose $n \geq 2$. To see that the family of transformations $X \mapsto U^* \overline{X} U$ is not superfluous in Theorem 10, we need to demonstrate two things:

1. that $X \mapsto U^* \overline{X} U$ is a Jordan automorphism, and
2. that no unitary $V \in \mathbb{C}^{n \times n}$ gives $\overline{X} = V^* X V$ for all X; otherwise, V could be absorbed into U.

Since entrywise conjugation is obviously invertible, the first amounts to showing that $\overline{A \circ B} = \overline{A} \circ \overline{B}$ for any $A, B \in \mathcal{H}^n(\mathbb{C})$ and that entrywise conjugation maps Hermitian matrices to Hermitian matrices. This is a routine computation.

For the second, let $n = 2$, and define the four Hermitian matrices,

$$E_1 \coloneqq \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ E_2 \coloneqq \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ E_3 \coloneqq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ E_4 \coloneqq \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.$$

If one assumes that $V^* E_i V = \overline{E_i}$ holds for $i \in \{1, 2, 3, 4\}$, a contradiction will become apparent. Thus when $n = 2$, the collection of transformations involving conjugation is not superfluous. When $n > 2$, extend the E_i with zeros. The same contradiction will manifest in the top-left 2×2 block.

Proposition 4. *The right-eigenvalues of a matrix $X \in \mathcal{H}^n(\mathbb{H})$ are the same as its Jordan-algebraic eigenvalues.*

Proof. Let E_i denote the matrix in $\mathcal{H}^n(\mathbb{H})$ having unity in the ith diagonal position and zeros elsewhere. The rank of $\mathcal{H}^n(\mathbb{H})$ as a Jordan algebra is n (Faraut and Korányi [5], Theorem V.3.7), and the set $\{E_1, E_2, \ldots, E_n\}$ consists of n idempotents such that $E_i \circ E_j = 0$ for $i \neq j$ and whose sum is I, making it a Jordan frame. Since $\mathcal{H}^n(\mathbb{H})$ is simple, there is by Theorem IV.2.5 of Faraut and Korányi some $\varphi \in \mathrm{JAut}(\mathcal{H}^n(\mathbb{H}))$ sending X to $\varphi(X) = \sum_{i=1}^{n} \lambda_i(X) E_i$, where the $\lambda_i(X)$ are its Jordan-algebraic eigenvalues. Noting the form of φ from Theorem 10, we see that we have diagonalized X as a matrix (Rodman [28], Theorem 5.3.6). Using Proposition 5.3.7 in Rodman one last time, those diagonal entries $\lambda_i(X)$ are its matrix right-eigenvalues. $\square$

Theorem 11. *If $m, n \in \mathbb{N}$ with $n \geq 3$, then*

1. $\mathrm{JAut}(\mathcal{L}^m)$ *is path-connected if $m \in \{0, 1\}$ and disconnected otherwise,*
2. $\mathrm{JAut}(\mathcal{H}^n(\mathbb{R}))$ *is path-connected if n is odd and disconnected otherwise,*
3. $\mathrm{JAut}(\mathcal{H}^n(\mathbb{C}))$ *is disconnected,*
4. $\mathrm{JAut}(\mathcal{H}^n(\mathbb{H}))$ *is path-connected,*
5. $\mathrm{JAut}(\mathcal{H}^3(\mathbb{O}))$ *is path-connected.*

Proof. In the matrix algebras $\mathcal{H}^n(\mathbb{A})$ where $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, we make the definition $\varphi_U \coloneqq X \mapsto U^* X U$ for $U \in \mathrm{Isom}(\mathbb{A}^n)$. Then if there is a path between U and V in $\mathrm{Isom}(\mathbb{A}^n)$, it can be used to construct a path between φ_U and φ_V in $\mathrm{JAut}(\mathcal{H}^n(\mathbb{A}))$. We proceed case-by-case, making use of Theorem 10.

1. After checking the two special cases, the first item follows from the isometry between $\mathrm{JAut}(\mathcal{L}^m)$ and $\mathrm{Isom}(\mathbb{R}^{m-1})$.
2. Let n be odd. A priori, $\mathrm{Isom}(\mathbb{R}^n)$ has two path components that correspond to the sign of the determinant. Suppose $U, V \in \mathrm{Isom}(\mathbb{R}^n)$ correspond to arbitrary $\varphi_U, \varphi_V \in \mathrm{JAut}(\mathcal{H}^n(\mathbb{R}))$. If $\det(U) = \det(V)$, then U, V belong to the same path component of $\mathrm{Isom}(\mathbb{R}^n)$, and we can construct a path from φ_U to φ_V using one from U to V. So, assume that $\det(U) = -\det(V)$. Since n is odd, we notice that $\det(U) = -\det(-U)$,

from which it follows that $\det(-V) = \det(U)$. There consequently exists a path between U and $-V$ in $\operatorname{Isom}(\mathbb{R}^n)$, and we can use it to construct a path from φ_U to φ_{-V} in $\operatorname{JAut}(\mathcal{H}^n(\mathbb{R}))$. Finally, since $\varphi_V = \varphi_{-V}$, we have in fact constructed a path from φ_U to φ_V.

Next let n be even. From the form of φ_U, it is clear that $\operatorname{JAut}(\mathcal{H}^n(\mathbb{R}))$ preserves the usual trace inner-product of real matrices. As a result, $\operatorname{JAut}(\mathcal{H}^n(\mathbb{R}))$ is topologically equivalent to a subset of $\operatorname{Isom}\left(\mathbb{R}^k\right)$ for $k = \left(n^2+n\right)/2 = \dim(\mathcal{H}^n(\mathbb{R}))$. Since the identity map has determinant one and is obviously a Jordan automorphism, we can show that $\operatorname{JAut}(\mathcal{H}^n(\mathbb{R}))$ is disconnected if we can find a $U \in \operatorname{Isom}(\mathbb{R}^n)$ such that $\det(\varphi_U) = -1$. The determinant of φ_U, as a real linear transformation, is the product of its complex eigenvalues. Since U^* is orthogonal, it is normal and has an orthonormal basis of complex eigenvectors $\{u_1, u_2, \ldots, u_n\}$ with corresponding complex eigenvalues λ_i. It's easy to check that

$$\left\{ u_i u_j^T + u_j u_i^T \;\middle|\; 1 \le j \le i \le n \right\}$$

is a basis for $\mathcal{H}^n(\mathbb{R})$ consisting of orthogonal complex eigenvectors of φ_U, each with complex eigenvalue $\lambda_i \lambda_j$. The determinant of φ_U is thus

$$\det(\varphi_U) = \prod_{1 \le j \le i \le n} \lambda_i \lambda_j = \prod_{1 \le i \le n} \lambda_i^{n+1}.$$

The second equality follows from the first by counting how many times each λ_i appears in the product: twice when $j = i$ and once for each $j \ne i$. From this expression it is clear that $U \coloneqq (-\operatorname{id}_{\mathbb{R}}) \times I$ will do the job.

3. We have expressed $\operatorname{JAut}(\mathcal{H}^n(\mathbb{C}))$ as the union of two sets. Suppose that $\varphi_U \coloneqq X \mapsto U^* X U$ belongs to the first, and $\psi_V \coloneqq X \mapsto V^* \overline{X} V$ to the second. If we set $\varphi_U = \psi_V$, then having $\overline{X} = (UV^*)^* X (UV^*)$ for all X would contradict Example 1, so the two sets must be disjoint. The maps $U \mapsto \varphi_U$ and $V \mapsto \psi_V$ are continuous; both sets are therefore a continuous image of the compact set $\operatorname{Isom}(\mathbb{C}^n)$. In particular, they are closed, and disjoint nonempty closed sets are separated.

4. The group $\operatorname{Isom}(\mathbb{H}^n)$ is path-connected (Tapp [31], Theorem 9.1).

5. F_4 is simply connected [33]. □

Combining Theorems 8 and 11 should answer any questions about the path-connectedness of $\operatorname{JAut}(V)$ when V is a Cartesian product or direct sum.

Example 2. It was once claimed that $\operatorname{JAut}\left(\mathcal{H}^3(\mathbb{O})\right)$ is disconnected [23], and the following counterexample (to the supposition that the identity component is the only component) was proposed:

$$\alpha\left(\begin{bmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{bmatrix}\right) \coloneqq \begin{bmatrix} \xi_1 & x_2 & \overline{x_3} \\ \overline{x_2} & \xi_3 & x_1 \\ x_3 & \overline{x_1} & \xi_2 \end{bmatrix}.$$

This, however, is not a Jordan automorphism. Recalling that $\{e_0, e_1, \ldots, e_7\}$ is our basis for $\mathbb{O}$ over $\mathbb{R}$, we have as a countercounterexample

$$\alpha\left(\begin{bmatrix} 0 & e_3 & -e_2 \\ -e_3 & 0 & 0 \\ e_2 & 0 & 0 \end{bmatrix}^2\right) \neq \alpha\left(\begin{bmatrix} 0 & e_3 & -e_2 \\ -e_3 & 0 & 0 \\ e_2 & 0 & 0 \end{bmatrix}\right)^2$$

because the $(2,3)$ entries differ on the left- and right-hand sides.

6 Derivative automorphisms

Our version of Ito and Lourenço's Theorem 5, specialized to a symmetric cone, is not much of a leap at this point.

Theorem 12. *Let $(V, \circ)$ be a Euclidean Jordan algebra of rank r with basis $\mathbf{b}$, $p \coloneqq \det_{\mathbf{b}}$, and $e \coloneqq \mathbf{b}(1_V)$. Then if $r \geq 4$ and if $i \in \{1, 2 \ldots, r-3\}$,*

$$\operatorname{Aut}\left(K_{p,e}^{(i)}\right) = \mathbb{R}_{++}\mathbf{b}\operatorname{JAut}(V)\,\mathbf{b}^{-1}.$$

Proof. Start from Theorem 5. In this case, $K_{p,e}$ is the cone of squares in V, so from Lemma 2 and Theorem 7,

$$\operatorname{Aut}(K_{p,e})_e = \operatorname{Aut}(\mathbf{b}(K))_{\mathbf{b}(1_V)} = \mathbf{b}\operatorname{Aut}(K)_{1_V}\mathbf{b}^{-1} = \mathbf{b}\operatorname{JAut}(V)\,\mathbf{b}^{-1}. \qquad \square$$

One would prefer if these derivatives were functions only of the cone. With respect to the direction $e = \mathbf{b}(1_V)$, the determinant polynomial is somewhat unique and could conceivably be omitted from the notation (Ito and Lourenço, Proposition 2.5). The direction $e = \mathbf{b}(1_V)$ however is essential if we want $\operatorname{JAut}(V)$ to arise. If you are desperate, it may be possible to change 1_V by changing the Jordan product (see Section III.3 of Faraut and Korányi [5]), but we don't find the possibility compelling enough to explore.

There's an obvious corollary to Theorem 12 for the simple algebras $\mathcal{H}^n(\mathbb{A})$ when $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. The case $\mathcal{H}^n(\mathbb{R})$ is Theorem 4.3 of Ito and Lourenço.

Corollary 2. *Suppose $V = \mathcal{H}^n(\mathbb{A})$ has basis $\mathbf{b}$, that $p = \det_{\mathbf{b}}$ and $e = \mathbf{b}(1_V)$, and that $n \geq 4$ and $i \in \{1, 2, \ldots, n-3\}$. If $\mathbb{A} \in \{\mathbb{R}, \mathbb{H}\}$ then*

$$\operatorname{Aut}\left(K_{p,e}^{(i)}\right) = \{X \mapsto \alpha U^* X U \mid \alpha > 0 \text{ and } U \in \operatorname{Isom}(\mathbb{A}^n)\}.$$

If instead $\mathbb{A} = \mathbb{C}$, then

$$\begin{aligned}\operatorname{Aut}\left(K_{p,e}^{(i)}\right) = &\{X \mapsto \alpha U^* X U \mid \alpha > 0 \text{ and } U \in \operatorname{Isom}(\mathbb{C}^n)\} \\ &\cup \left\{X \mapsto \alpha U^* \overline{X} U \;\middle|\; \alpha > 0 \text{ and } U \in \operatorname{Isom}(\mathbb{C}^n)\right\}.\end{aligned}$$

The Jordan spin algebras and the Albert algebra, of rank two and three respectively, are absent from Corollary 2 because Theorem 12 requires rank four or more. They can however show up as factors in a Cartesian product or direct sum. This is most convenient when the standard basis is used for the Jordan spin algebra, because in that case the basis representation map is the identity. For example, the algebra

$$V \coloneqq \underbrace{\mathcal{L}^1 \times \mathcal{L}^1 \times \cdots \times \mathcal{L}^1}_{n \text{ times}}$$

has componentwise real-number multiplication (the Hadamard product) for its Jordan product, and $\operatorname{JAut}\left(\mathcal{L}^1\right) = \{\mathrm{id}_{\mathbb{R}}\}$ on each factor. Using Theorems 8 and 12 with the standard basis, one can easily find the automorphisms with respect to the determinant polynomial $X_1 X_2 \cdots X_n$ in direction $1_V = (1, 1, \ldots, 1)$. This was Theorem 4.1 of Ito and Lourenço [16].

7 Conclusions

Motivated by the Ito & Lourenço result,

$$\operatorname{Aut}\left(K_{p,e}^{(i)}\right) = \mathbb{R}_{++} \operatorname{Aut}\left(K_{p,e}\right)_e, \tag{2}$$

and with symmetric cones in mind, we first provided a new proof of the identity

$$\operatorname{JAut}(V) = \operatorname{Aut}(K)_{1_V} \tag{3}$$

where V is a Euclidean Jordan algebra and K its cone of squares. Our proof used the famous decomposition of V into simple components on which familiar results of Faraut and Korányi can be cited. We also showed in a novel way how $\operatorname{JAut}(V)$ decomposes into the Jordan-automorphism groups of simple components. What are those groups? (There should be only five types.) To answer that question, we answered a harder one: what are the automorphism groups of the irreducible symmetric cones? Our main contribution there was to find $\operatorname{Aut}\left(\mathcal{H}_+^n(\mathbb{H})\right)$.

We then substituted those cone automorphisms into Equation (3) to elicit the corresponding Jordan-automorphism groups. The resulting explicit description of $\operatorname{JAut}(\mathcal{H}^n(\mathbb{H}))$ was new, and we were able to correct an existing result for $\mathcal{H}^n(\mathbb{C})$. We also analyzed the path-connectedness of those Jordan-automorphism groups for what we hope is the first time.

Finally, we assembled Equations (2) and (3) to obtain Theorem 12, showing that, in coordinates, the derivative automorphisms of a symmetric cone are positive multiples of Jordan automorphisms in the associated Euclidean Jordan algebra. Having already listed the Jordan automorphisms of the simple algebras, this result was easily specialized to the derivatives of the positive-semidefinite cones in Corollary 2, completing our mission.

Two pieces remain missing from the puzzle. We clearly cheated when describing $\operatorname{JAut}\left(\mathcal{H}^3(\mathbb{O})\right)$ as the Lie group F_4. That's nice, but what is F_4? Baez [2] and Yokota [33] define it to be $\operatorname{JAut}\left(\mathcal{H}^3(\mathbb{O})\right)$! We believe this to be answerable, but not without a significant detour, so we have let it lie for the time being along with any questions about $\operatorname{Aut}\left(\mathcal{H}_+^3(\mathbb{O})\right)$.

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