Clans and homogeneous cones

Michael Orlitzky

National Taiwan Normal University
June 20th, 2025

Introduction

Definition.

A *homogeneous* cone K is a proper cone whose automorphism group acts transitively on its interior:

$$\operatorname{Aut}(K)\, e = K \text{ for all } e \in \operatorname{int}(K)$$

Example.

All symmetric cones are homogeneous:

1. Second-order cones $\mathcal{L}^n_+$
2. Real PSD cones $\mathcal{H}^n_+(\mathbb{R})$
3. Complex PSD cones $\mathcal{H}^n_+(\mathbb{C})$
4. Quaternion PSD cones $\mathcal{H}^n_+(\mathbb{H})$
5. The 3×3 Octonion "PSD" cone $\mathcal{H}^3_+(\mathbb{O})$

Example (dual Vinberg cone).

…but not all homogeneous cones are symmetric.

$$K := \left\{ x \in \mathbb{R}^5 \;\middle|\; \begin{array}{c} x_1, x_2 \geq 0 \\ x_1 x_2 x_3 - x_1 x_5^2 - x_2 x_4^2 \geq 0 \end{array} \right\}$$

is homogeneous but not self-dual.

Theorem (Vinberg 1963, well-known).

There is a one-to-one correspondence between homogeneous cones and T-algebras.

Theorem (Vinberg 1963, less well-known).

There is a one-to-one correspondence between homogeneous cones and clans with unit elements.

Question.

So what is a *clan*?

Definition.

A *clan* is an algebra $\mathcal{C}$ with multiplication Δ such that the left regular representation $L_x \coloneqq x \mapsto x \Delta y$ satisfies

- $L_x L_y - L_y L_x = L_{(x \Delta y - y \Delta x)}$
- The eigenvalues of L_x are real for all $x \in \mathcal{C}$
- $(x, y) \mapsto \operatorname{trace}\left(L_{x \Delta y}\right)$ is an inner product

Building a clan

Theorem (Vinberg 1963).

If K is homogeneous, then for any $e \in \operatorname{int}(K)$ we have $\operatorname{Aut}(K) = \operatorname{Aut}(K)_e \, T(K)$ where

- $\operatorname{Aut}(K)_e$ is the stabilizer subgroup of e
- $T(K)$ is,
  - simply transitive
  - maximally connected
  - triangular
- $\operatorname{Aut}(K)_e \cap T(K) = \{\mathrm{id}\}$
- $\operatorname{Aut}(K)_e$ and $T(K)$ are Lie subgroups

Definition.

A *triangular* group is a group whose matrix representations are triangular in some basis.

Definition.

$T(K)$ being *simply transitive* on K means that for all $e, x \in \operatorname{int}(K)$, there exists a **unique** $A \in T(K)$ such that $Ae = x$.

$T(K)$ is a Lie group which lets us define

$$\mathfrak{t}(K) \coloneqq \operatorname{Lie}(T(K)) \coloneqq \{X : V \to V \mid \exp(tX) \in T(K) \text{ for all } t \in \mathbb{R}\}$$

Here and from now on, $V \coloneqq \operatorname{span}(K)$ is the ambient vector space.

Note.

$\mathfrak{t}(K)$ is itself triangular by the Lie group/algebra correspondence.

Lemma.

The map $\exp : \mathfrak{t}(K) \to T(K)$ is a diffeomorphism.

Proof. The book of Gorbatsevich, Onishchik, and Vinberg[1] shows that $T(K)$ is simply connected.

If $X \in \mathfrak{t}(K)$, then both X and X^T are triangular. The eigenvalues of $\operatorname{ad}_X(Y) \coloneqq XY - YX$ are all real—we can find them. This is equivalent to the lemma by a famous result in the same book. □

1 Lie Groups and Lie Algebras III

Corollary.

$\operatorname{int}(K)$, $T(K)$, and $\mathfrak{t}(K)$ are all in bijection.

Proof.

Fix $e \in \operatorname{int}(K)$ and associate to $x \in \operatorname{int}(K)$ the unique $A \in T(K)$ such that $Ae = x$. By the previous lemma, the exponential is a bijection $\mathfrak{t}(K) \to T(K)$. □

Theorem (Vinberg 1963).

There is a unital clan $\mathcal{C}_{K,e}$ associated with every homogeneous cone K and point $e \in \operatorname{int}(K)$.

Proof.

Since $T(K)$ is a Lie group, the map

$$f : T(K) \to \operatorname{int}(K)$$
$$f = A \mapsto Ae$$

is differentiable with $df_X \approx f$ at every $X \in T(K)$.

Proof (cont'd).

The tangent planes to $T(K)$ are $\mathfrak{t}(K)$, and the tangent planes to $\operatorname{int}(K)$ are the ambient space V.

It follows that the differentials

$$df_X : \mathfrak{t}(K) \to V$$
$$df_X = A \mapsto Ae$$

are all equivalent to df_{id}.

Proof (cont'd).

If df_{id} is singular, df_X is singular for all X.

In that case, Sard's theorem would say that $f(T(K))$ has measure zero in $\operatorname{int}(K)$. But $f(T(K)) = \operatorname{int}(K)$ by simple transitivity, so this is not the case.

Thus, df_{id} is invertible.

Proof (cont'd).

This allows is to define linear operators,

$$L_x \coloneqq df_{\mathrm{id}}^{-1}(x)$$

which, for $x, y \in V$, let us define multiplication:

$$x \Delta y \coloneqq L_x y$$

This product is bilinear, so V becomes an algebra.

Proof (cont'd).

Recall that $df_{\mathrm{id}}(A) = Ae$. By definition

$$y \Delta e = L_y e = \left[df_{\mathrm{id}}^{-1}(y) \right] e,$$

where $df_{\mathrm{id}}^{-1}(y) \in \mathfrak{k}(K)$ sends e to y. It follows that e is a right unit element.

The element L_z that sends e to e is unique, and $z = e$ works. Thus e is a left unit as well.

Proof (cont'd).

The map

$$(x, y) \mapsto \operatorname{trace}\left(L_{x\Delta y}\right)$$

is bilinear, symmetric, and positive-definite. As a result we may define an inner product on V by

$$\langle x, y\rangle \coloneqq \operatorname{trace}\left(L_{x\Delta y}\right).$$

Under this inner product, V becomes a *clan*. We denote it by $\mathcal{C}_{K,e}$. □

Clan-based results

Proposition.

If $A \in \operatorname{Aut}(\mathcal{C}_{K,e})$, then A is a clan isometry.

Proof.

The claim is that $L_{x\Delta y}$ and $L_{Ax\Delta Ay}$ have the same trace. We see from the definition of a clan automorphism that that eigenvectors z of $L_{x\Delta y}$ correspond (with the same eigenvalue) to eigenvectors Az of $L_{Ax\Delta Ay}$.

It follows that the two share a spectrum. □

Proposition.

If $A \in \operatorname{Aut}(\mathcal{C}_{K,e})$, then $A \in \operatorname{Aut}(K)$.

Proof.

Suppose A is a clan automorphism, but not a cone automorphism. Then $Ax \notin \operatorname{int}(K)$ for some $x \in \operatorname{int}(K)$. We will see that this is not possible.

We begin by noticing that $L_{Az} = AL_zA^{-1}$ since A is a clan automorphism. As a result…

Proof (cont'd).

$$\exp\left(L_{Az}\right) = A \exp\left(L_z\right) A^{-1} \in T\left(K\right)$$

Recalling that $V \overset{df_{\mathrm{id}}}{\leftrightarrow} \mathfrak{t}\left(K\right) \overset{\exp}{\leftrightarrow} T\left(K\right)$ are in bijection and that $T\left(K\right)$ is simply transitive, we let $z \in V$ be such that $\exp\left(L_z\right)$ sends e to x. Then since $A\left(e\right) = e$,

$$A \exp\left(L_z\right) A^{-1} e = A \exp\left(L_z\right) e = Ax \notin \operatorname{int}\left(K\right)$$

This would contradict $\exp\left(L_{Az}\right)$ being in $T\left(K\right)$. $\square$

Proposition.

If $A \in \operatorname{Aut}(K)_e$ then $A \in \operatorname{Aut}(\mathcal{C}_{K,e})$.

Proof (pt. 2).

Suppose for now that $AL_xA^{-1} \in \mathfrak{t}(K)$ for all $x \in V$. In that case $(A^{-1}L_{Ax}A)\,e = x$, and by uniqueness,

$$A^{-1}L_{Ax}A = L_x \iff L_{Ax} = AL_xA^{-1}$$

It follows that

$$A(x \Delta y) = (AL_xA^{-1})(Ay) = L_{Ax}(Ay) = Ax \Delta Ay$$

Proof (pt. 1).

It remains to show that $AL_x A^{-1}$ belongs to $\mathfrak{t}(K)$ for all $x \in V$. We obtain an equivalent problem via exp: does $\exp(AL_x A^{-1}) = A \exp(L_x) A^{-1}$ belong to $T(K)$?

Recall that $\operatorname{Aut}(K) = \operatorname{Aut}(K)_e T(K)$. Conjugation by $A \in \operatorname{Aut}(K)_e$ trivially preserves $\operatorname{Aut}(K)$ and $\operatorname{Aut}(K)_e$ so it must preserve $T(K)$ as well. $\square$

Lemma (Hofmann and Terp 1994).

If K is proper, there exist maximal compact subgroups of $\operatorname{Aut}(K)$, and all are of the form $\operatorname{Aut}(K)_e$ for some $e \in \operatorname{int}(K)$.

Corollary (cf. Faraut and Korányi I.1.8).

If K is homogeneous and if $e \in \operatorname{int}(K)$, then $\operatorname{Aut}(K)_e$ is a maximal compact subgroup of $\operatorname{Aut}(K)$.

Theorem.

If K is a homogeneous cone and if $e \in \operatorname{int}(K)$, then

$$\operatorname{Aut}(\mathcal{C}_{K,e}) = \operatorname{Aut}(K)_e = \operatorname{Aut}(K) \cap \operatorname{Isom}(\mathcal{C}_{K,e})$$

Proof.

$\operatorname{Aut}(K) \cap \operatorname{Isom}(\mathcal{C}_{K,e})$ is compact subgroup of $\operatorname{Aut}(K)$. The maximality of $\operatorname{Aut}(K)_e$ therefore provides the missing inclusion. □

The end