Isometric automorphism identities for symmetric, homogeneous, and hyperbolicity cones

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conto2025, Kyoto
December 2nd

Motivation

Every symmetric cone arises as the cone of squares in some Euclidean Jordan algebra.

Examples:

- The nonnegative orthant
- The PSD cone
- The second-order (Lorentz, ice-cream) cone

A Euclidean Jordan algebra is,

- A finite-dimensional real vector space V
- A commutative bilinear multiplication $x \circ y$
- A unit element e
- An "associative" inner product

Within V there is a (symmetric) *cone of squares*,

$$K := \{x \circ x \mid x \in V\}$$

Theorem (Orlitzky[†] 2025).

$$\operatorname{Aut}(K)_e = \operatorname{Aut}(K) \cap \operatorname{Isom}(V)$$

if and only if every algebra isomorphism between simple factors of V is an isometry.

Corollary (cf. Faraut and Korányi, p.57).

$$\operatorname{Aut}(\circ) = \operatorname{Aut}(K)_e = \operatorname{Aut}(K) \cap \operatorname{Isom}(V)$$

under either of the usual trace inner products.

[†] Jordan and isometric cone automorphisms in Euclidean Jordan algebras

Euclidean Jordan algebras have a fixed-size spectral decomposition with real eigenvalues, and K is the set where those eigenvalues are nonnegative.

Corollary.

If $\operatorname{Aut}(\lambda)$ denotes the spectrum-preserving maps, then

$$\operatorname{Aut}(\lambda) = \operatorname{Aut}(\circ) = \operatorname{Aut}(K)_e$$

and, under the conditions of the theorem,

$$\operatorname{Aut}(\lambda) = \operatorname{Aut}(K) \cap \operatorname{Isom}(V).$$

In conic optimization, symmetric cones live at the bottom of a hierarchy[†]:

$$\text{symmetric} \subsetneq \text{homogeneous} \subsetneq \text{hyperbolicity}.$$

As we go up, the cones get harder to work with.

Question.

How far up can we extend the characterization of isometric cone automorphisms?

[†] Refined significantly by Gouveia, Ito, Lourenço, Roschina, and Saunderson

Hyperbolicity cones

The pair (p, e) is *hyperbolic* in $\mathbb{R}^n$ if

1. $p : \mathbb{R}^n \to \mathbb{R}$ is a homogeneous polynomial
2. $e \in \mathbb{R}^n$ and $p(e) > 0$
3. the roots of $\lambda \mapsto p(\lambda e - x)$ are real for all $x \in \mathbb{R}^n$

Example. The pair $(\det, I)$ is hyperbolic in $\mathcal{S}^n$.

We can express det as a homogeneous polynomial of degree n on $\mathcal{S}^n$, and the roots of $\lambda \mapsto \det(\lambda I - X)$ are the (all real) eigenvalues of X.

We write $\lambda_{p,e}(x)$ for the real roots of $\lambda \mapsto p(\lambda e - x)$, and call them the eigenvalues of x.

Definition.

The *hyperbolicity cone* of (p, e) is,

$$\Lambda_{p,e} \coloneqq \left\{ x \in \mathbb{R}^n \,\middle|\, \lambda_{p,e}(x) \geq 0 \right\}.$$

It is always closed, convex, and solid.

The pair (p, e) associated with $\Lambda_{p,e}$ is not unique:

- If $\tilde{e} \in \operatorname{int}\left(\Lambda_{p,e}\right)$, then $\Lambda_{p,\tilde{e}} = \Lambda_{p,e}$
- Squaring p, for example, does not affect $\Lambda_{p,e}$

Definition.

p is *minimal* if, among all polynomials q such that $\Lambda_{q,e} = \Lambda_{p,e}$, the degree of p is minimal.

How to characterize $\operatorname{Aut}\left(\Lambda_{p,e}\right)$?

We no longer have access to algebra automorphisms, but we do have spectrum-preserving maps[†].

Definition.

A is an *automorphism of* (p, e) if A is invertible and if $\lambda_{p,e}(Ax) = \lambda_{p,e}(x)$ for all $x \in \mathbb{R}^n$.

The group of all such automorphisms is $\operatorname{Aut}(p, e)$.

† Motivated by results of Gowda, Jeong, and Sossa for semi-FTvN systems

Theorem.

If $\Lambda_{p,e}$ is pointed and if p is minimal, then

$$\operatorname{Aut}(p, e) = \operatorname{Aut}\left(\Lambda_{p,e}\right)_{e}.$$

Proof. The easy inclusion is easy.

The other uses the 2023 characterization of $\operatorname{Aut}\left(\Lambda_{p,e}\right)$ due to Ito and Lourenço[†] for minimal p. $\square$

† Automorphisms of rank-one generated hyperbolicity cones and their derivative relaxations

Definition (Baushke et al.[†] 2001).

If $\Lambda_{p,e}$ is pointed, then

$$\langle x, y \rangle_{p,e} := \frac{1}{4} \left\| \lambda_{p,e}(x+y) \right\|_2^2 - \frac{1}{4} \left\| \lambda_{p,e}(x-y) \right\|_2^2 .$$

is an inner product. We let $\operatorname{Isom}(p, e)$ denote the corresponding isometry group.

Proposition (obvious). If p is minimal, then

$$\operatorname{Aut}(p, e) = \operatorname{Aut}\left(\Lambda_{p,e}\right)_e \subseteq \operatorname{Isom}(p, e) .$$

† Hyperbolic Polynomials and Convex Analysis

We are missing only one inclusion,

$$\operatorname{Isom}(p, e) \subseteq \operatorname{Aut}(p, e) = \operatorname{Aut}\left(\Lambda_{p,e}\right)_e .$$

Towards that end:

Lemma (Hofmann and Terp[†] 1994).

If K is a proper cone, there exist maximal compact subgroups of $\operatorname{Aut}(K)$, and all of its maximal compact subgroups are of the form $\operatorname{Aut}(K)_e$ for $e \in \operatorname{int}(K)$.

[†] Compact Subgroups of Lie Groups and Locally Compact Groups

Theorem.

If $\Lambda_{p,e}$ is pointed, there exists a pair $(\tilde{p}, \tilde{e})$ with the same hyperbolicity cone that satisfies

$$\operatorname{Aut}(\tilde{p}, \tilde{e}) = \operatorname{Aut}\left(\Lambda_{\tilde{p},\tilde{e}}\right)_{\tilde{e}} = \operatorname{Aut}\left(\Lambda_{\tilde{p},\tilde{e}}\right) \cap \operatorname{Isom}(\tilde{p}, \tilde{e}).$$

Proof.

Choose $\tilde{p}$ minimal for inclusion; then if $\operatorname{Aut}\left(\Lambda_{\tilde{p},\tilde{e}}\right)_{\tilde{e}}$ is chosen maximal, the inclusion cannot be strict. $\square$

Compared to the result for Euclidean Jordan algebras, this has several drawbacks; namely, it requires:

- A specific inner product
- An unspecified interior point $\tilde{e}$
- A minimal polynomial

This improves slightly for **homogeneous cones**.

A *homogeneous cone* is a proper cone whose automorphism group acts transitively on its interior.

Theorem.

If $\Lambda_{p,e}$ is homogeneous and if p is minimal, then

$$\operatorname{Aut}(p, e) = \operatorname{Aut}\left(\Lambda_{p,e}\right)_e = \operatorname{Aut}\left(\Lambda_{p,e}\right) \cap \operatorname{Isom}(p, e).$$

In this case, the point e does not need to be changed because all interior points are equivalent up to cone automorphisms.

Question.

How do we find a minimal polynomial for a homogeneous cone?

Answer.

Nobody knows. We will see in a moment that a generic approach is not promising.

Polynomials for homogeneous cones

Theorem (Vinberg[†] 1963).

The following are (up to isomorphism) in one-to-one correspondence:

$$\begin{array}{rcl} \text{homogeneous cones} & \iff & \text{clans with unity} \\ & \iff & \\ \text{N-algebras} & \iff & \text{T-algebras} \end{array}$$

[†] The theory of convex homogeneous cones

The elements of a T-algebra are matrices whose entries come from real, non-associative algebras of various sizes—though the diagonals remain 1×1.

In a T-algebra, Vinberg shows that the set

$$K \coloneqq \left\{ tt^{*} \;\middle|\; \begin{array}{c} t \text{ is upper-triangular with} \\ \text{positive diagonal entries} \end{array} \right\}$$

is a homogeneous cone, and that all homogeneous cones arise in this manner.

The decomposition $x = t(x)\, t(x)^*$ is unique, and hints at a T-algebraic definition of eigenvalues.

Let X be a real, symmetric, positive-definite matrix. Then X has a Cholesky decomposition $X = TT^*$ for which

$$\det(X) = \det(T)^2 = \prod_{i=1}^{r} \lambda_i(T)^2 = \prod_{i=1}^{r} T_{ii}^2.$$

We have access to the $t(x)_{ii}$ in a T-algebra, so…

In 1997, Güler[†] defined

$$p(x) := \prod_{i=1}^{r} t(x)_{ii}^{2d_i},$$

and showed that, if the $d_i \in \mathbb{N}$ are chosen carefully, then p has the desired properties.

But unfortunately, the d_i grow quite large, making p non-minimal in most examples.

† Hyperbolic Polynomials and Interior Point Methods for Convex Programming

Theorem (Gouveia, Ito, and Lourenço[†] 2025).

In a T-algebra of rank r, degree 2^{r-1} is sufficient.

Sufficiency is implied by the *composite determinant* D_r of degree 2^{r-1}.

Theorem (Nakashima[‡] 2024).

Degree 2^{r-1} can be necessary.

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† Faces of homogeneous cones and applications to homogeneous chordality

‡ An example of homogeneous cones whose basic relative invariant has maximal degree

Proposition.

Güler's polynomial is the composite determinant D_r of degree 2^{r-1}.

Proof.

The polynomials $D_1, \ldots, D_r$ are defined recursively by,

$$D_1(x) \coloneqq x_{11}$$
$$D_i(x) \coloneqq t(x)_{ii}^2 \prod_{j<i} D_j(x).$$

Proof (cont'd).

If you write

$$t(x)_{ii}^{2} = \frac{D_i(x)}{\prod_{j<i} D_j(x)}$$

and try to cancel the denominators by playing with the exponents d_i on $t(x)_{ii}^{2d_i}$, a pattern will become clear. Or you can just check:

$$p(x) = t(x)_{11}^{2(r-2)} \cdots t(x)_{(r-1)(r-1)}^{2} \, t(x)_{rr}^{2} = D_r(x).$$

Alas, this does not help: D_r is rarely minimal, and Ito and Lourenço[†] have an example showing that

$$\mathrm{Aut}\left(\Lambda_{p,e}\right)_e \nsubseteq \mathrm{Aut}\left(p, e\right)$$

in general when p is not minimal.

Conclusion.

Degree 2^{r-1} is still too large for most examples. A generic construction may be asking too much.

[†] Automorphisms of rank-one generated hyperbolicity cones and their derivative relaxations

> *…it may be more advantageous to deal with a homogeneous cone…on its own terms…instead of seeing it as a hyperbolicity cone or as a slice of a positive semidefinite cone.*
>
> *–Gouveia, Ito, and Lourenço*

Question.

But what should we use instead?

Option 1. Clans with unity?

Good:

- No funny business with the basis
- Algebra structure is on the original space
- Bijections between the space, cone, transitive automorphisms, and their Lie algebra

Bad:

- Not as popular as T-algebras

Option 2. T-algebras?

Good:

- Well-studied
- Eigenvalues from the Cholesky factorization
- Facial structure (e.g. extreme rays) known

Bad:

- Involves a change of basis
- Have “extra” dimensions

Clans (briefly)

Definition.

A unital *clan* $\mathcal{C}$ is an algebra on V associated with a homogeneous cone $K \subseteq V$ and $e \in \operatorname{int}(K)$…

- Multiplication $x \Delta y$
- Unit element e
- Left-regular representation $L_x \coloneqq y \mapsto x \Delta y$ whose eigenvalues are always real
- Inner product $(x, y) \mapsto \operatorname{trace}\left(L_{x \Delta y}\right)$

Theorem (Vinberg 1963).

If K is homogeneous, then for any $e \in \operatorname{int}(K)$ we have $\operatorname{Aut}(K) = \operatorname{Aut}(K)_e\, T(K)$ where $T(K)$ is

- Simply transitive
- Maximally connected
- Triangular

Being *simply transitive* puts $T(K)$ in bijection with $\operatorname{int}(K)$ via $g \mapsto ge$.

In fact, by Sard's theorem and some Lie theory, the following are all in bijection:

$$\mathrm{int}(K) \leftrightarrow T(K) \leftrightarrow \mathrm{Lie}(T(K)) \leftrightarrow V.$$

The clan multiplication is defined in terms of this bijection. If L_x is the element of $\mathrm{Lie}(T(K))$ corresponding to $x \in V$, then

$$x \Delta y \coloneqq L_x y.$$

Proposition (Kai and Nomura[†] 2005).

If $A \in \operatorname{Aut}(\Delta)$, then A is a clan isometry.

Lemma (trivial).

If $A \in \operatorname{Aut}(\Delta)$, then $Ae = A^{-1}e = e$, and $L_{Ax} = AL_xA^{-1}$ for all x.

† A characterization of symmetric cones through pseudoinverse maps

Proposition.

If $A \in \operatorname{Aut}(\Delta)$, then $A \in \operatorname{Aut}(K)$.

Proof.

If A is in $\operatorname{Aut}(\Delta)$ but not $\operatorname{Aut}(K)$, then WLOG, A sends some $x \in \operatorname{int}(K)$ to $y \notin \operatorname{int}(K)$. Use bijectivity to find the $z \in V$ such that $\exp(L_z)\, e = x$. Then,

$$\exp(L_{Az}) = \exp\left(AL_zA^{-1}\right) = A \exp(L_z)\, A^{-1}$$

satisfies $\exp(L_{Az})\, e = Ax = y$. Contradiction. $\square$

Thus:

$$\operatorname{Aut}(\Delta) \subseteq \begin{cases} \operatorname{Aut}(K)_e \\ \operatorname{Aut}(K) \cap \operatorname{Isom}(\mathcal{C}) \end{cases}$$

But reverse inclusions appear false!

Question.

How are $\operatorname{Aut}(K)_e$ and $\operatorname{Aut}(K) \cap \operatorname{Isom}(\mathcal{C})$ related?

T-algebras (briefly)

Definition.

A T-algebra $\mathcal{A}$ of rank r is a generalized matrix space,

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_{11} & \cdots & \mathcal{A}_{1r} \\ \vdots & \ddots & \vdots \\ \mathcal{A}_{r1} & \cdots & \mathcal{A}_{rr} \end{bmatrix}$$

where

- Diagonal components are isomorphic to $\mathbb{R}$
- $(x, y) \mapsto \operatorname{trace}(xy^*)$ is an inner product
- Lots of other things

Recall the homogeneous cone assosociated with $\mathcal{A}$,

$$K := \left\{ tt^{*} \;\middle|\; \begin{array}{c} t \in \mathcal{A} \text{ is upper-triangular and} \\ \text{has positive diagonal entries} \end{array} \right\}.$$

Any $x \in K$ maps to $t(x)$ uniquely. (See Gouveia, Ito, and Lourenço[†] for the boundary.)

Warning.

Components of $\mathcal{A}$ are allowed to be trivial—this can break your matrix intuition.

† Faces of homogeneous cones and applications to homogeneous chordality

Proposition.

If $A \in \operatorname{Aut}(\mathcal{A})$, then $A \in \operatorname{Isom}(\mathcal{A})$.

Proof.

The definition of T-algebra automorphism implies that A, at worst, permutes the diagonals, thereby preserving the trace. And

$$\langle Ax, Ax \rangle \coloneqq \operatorname{trace}\left(A(x)\,A(y)^{*}\right) = \langle x, y \rangle$$

as A preserves multiplication and involution. $\square$

Proposition.

If $A \in \mathrm{Aut}(\mathcal{A})$, then $A \in \mathrm{Aut}(K)$.

Proof.

T-algebra automorphisms preserve the triangular subspaces. If $x = tt^* \in K$, then

$$A(x) = A(tt^*) = A(t)A(t)^*$$

where $A(t)$ is upper-triangular and has the same positive diagonals as t. Likewise for A^{-1}. □

Thus:

$$\operatorname{Aut}(\mathcal{A}) \subseteq \begin{cases} \operatorname{Aut}(K)_e \\ \operatorname{Aut}(K) \cap \operatorname{Isom}(\mathcal{A}) \end{cases}$$

But reverse inclusions appear false!

Question.

How are $\operatorname{Aut}(K)_e$ and $\operatorname{Aut}(K) \cap \operatorname{Isom}(\mathcal{A})$ related?

Conclusions and future work

Recap:

- Tried to use clan/T-algebra automorphisms to avoid minimality, but they're too strict.
- Trace isometries in either case are still open to characterization.
- Have not invoked the spectrum so far. Spectral preservers and isometries are also open to characterization.

Lots of low-hanging fruit with the spectrum:

- What are the Güler eigenvalues?
- Can we define eigenvalues directly, without worrying about polynomials?
- What should the spectral preservers be?
- Can we choose a spectral norm consistently?

In a Euclidean Jordan algebra, eigenvalues are roots of a generic characteristic polynomial†. This only works in a power-associative algebra, however.

- The upper-triangular part of a T-algebra is associative. If we mimic this process, do the diagonals come out as the eigenvalues?
- If not, what do we get? Are *they* polynomials in the entries of x?
- What preserves them? Et cetera.

† The characteristic polynomial and determinant are not ad hoc constructions

Option 3. Connectedness algebras?

There is a *third* product on the Hermitian part of $\mathcal{A}$:

$$x \square y \coloneqq \frac{xy + yx}{2}.$$

Here we know that $\operatorname{Aut}(K)_e \subseteq \operatorname{Aut}(\square)$, and it is the reverse inclusion $\operatorname{Aut}(\square) \subseteq \operatorname{Aut}(K)$ that is missing.

- Works in an EJA because $K = \{x \square x \mid x \in \mathcal{A}\}$
- Third inclusion $\operatorname{Aut}(\square) \subseteq \operatorname{Isom}(\mathcal{A})$ likely holds if we need it

The end