Jordan/isometric cone automorphisms in Euclidean Jordan algebras

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Motivation

Example.

The real positive-semidefinite cone:

$$\mathcal{H}^{n}_{+}(\mathbb{R}) \coloneqq n \times n \text{ real, symmetric, PSD matrices}$$

This cone…

- is symmetric (self-dual, homogeneous)
- has applications everywhere
- is inherently interesting if you attend ILAS

Its automorphisms are known:

$$\operatorname{Aut}\left(\mathcal{H}_{+}^{n}\left(\mathbb{R}\right)\right) = \left\{X \mapsto U^{*}XU \mid U \in \mathrm{GL}_{n}\left(\mathbb{R}\right)\right\}$$

This automorphism group…

- preserves nonnegativity of eigenvalues
- in fact, preserves “inertia”
- fails to preserve the identity
- fails to preserve multiplication

The vector space

$$\mathcal{H}^n(\mathbb{R}) \coloneqq n \times n \text{ real symmetric matrices}$$

becomes an *algebra* with unit element I (the identity matrix) under the multiplication,

$$X \circ Y \coloneqq \frac{1}{2}(XY + YX)$$

This algebra has the usual inner product,

$$\langle X, Y \rangle \coloneqq \operatorname{trace}(XY)$$

The *cone of squares* in $\mathcal{H}^n(\mathbb{R})$ is,

$$\left\{ X \circ X = XX \mid X \in \mathcal{H}^n(\mathbb{R}) \right\} = \mathcal{H}^n_+(\mathbb{R})$$

It is well-known that this is the subset where all eigenvalues are nonnegative.

Claim.

The isometries of $\mathcal{H}^n(\mathbb{R})$, automorphisms of $\mathcal{H}^n(\mathbb{R})$, and automorphisms of $\mathcal{H}^n_+(\mathbb{R})$ are all closely related.

It turns out that

$$\operatorname{JAut}\left(\mathcal{H}^{n}\left(\mathbb{R}\right)\right) = \operatorname{Aut}\left(\mathcal{H}^{n}_{+}\left(\mathbb{R}\right)\right)_{I}$$

$$=$$

$$\operatorname{Aut}\left(\mathcal{H}^{n}_{+}\left(\mathbb{R}\right)\right) \cap \operatorname{Isom}\left(\mathcal{H}^{n}\left(\mathbb{R}\right)\right)$$

where

- JAut is an *algebra* automorphism group
- G_I denotes a stabilizer subgroup of G
- Isom is an isometry group

Goal.

Characterize the scenarios where

$$\operatorname{JAut}(V) = \operatorname{Aut}(K)_e = \operatorname{Aut}(K) \cap \operatorname{Isom}(V)$$

holds in a more general setting:

- V an algebra
- K a cone in V
- $e \in K$
- $\operatorname{Isom}(V)$ with respect to some inner product

Euclidean Jordan algebras

Observation.

$\mathcal{H}_{+}^{n}(\mathbb{R})$ is symmetric and irreducible.

Theorem.

Every symmetric cone is the cone of squares in some Euclidean Jordan algebra (EJA). Irreducible cones correspond to simple algebras.

Definition (Euclidean Jordan algebra).

A *Euclidean Jordan algebra* (EJA) is,

- a finite-dimensional real vector space V
- a commutative bilinear multiplication $x \circ y$
- a unit element e
- an inner product with $\langle x \circ y, z \rangle = \langle x, y \circ z \rangle$

Within V we have the *cone of squares*,

$$K \coloneqq \left\{ x \circ x \,\middle|\, x \in V \right\}$$

Example.

Several popular cones arise in this manner:

1. Second-order cones $\mathcal{L}^n_+$
2. Real PSD cones $\mathcal{H}^n_+(\mathbb{R})$
3. Complex PSD cones $\mathcal{H}^n_+(\mathbb{C})$
4. Quaternion PSD cones $\mathcal{H}^n_+(\mathbb{H})$
5. The 3×3 Octonion "PSD" cone $\mathcal{H}^3_+(\mathbb{O})$

Definition (EJA spectral decomposition).

Every element x in a Euclidean Jordan algebra has a set of Jordan-algebraic eigenvalues from its spectral decomposition,

$$x = \lambda_1(x) e_1 + \lambda_2(x) e_2 + \cdots + \lambda_r(x) e_r$$

Note.

The cone of squares is the set of x where $\lambda(x) \geq 0$.

Definition (EJA trace).

The Jordan-algebraic trace of an element is the sum of its eigenvalues,

$$\operatorname{tr}(x) \coloneqq \lambda_1(x) + \lambda_2(x) + \cdots + \lambda_r(x)$$

This induces an inner product on any EJA:

$$\langle x, y \rangle_{\operatorname{tr}} \coloneqq \operatorname{tr}(x \circ y)$$

Definition (left regular representation).

The "left multiplication by" operator induces its own inner product on an EJA,

$$L_x \coloneqq y \mapsto x \circ y$$
$$\langle x, y \rangle_L \coloneqq \operatorname{trace}\left(L_{x \circ y}\right)$$

Note.

This trace is the usual trace of a linear operator.

Theorem (known).

$$\operatorname{JAut}(V) = \operatorname{Aut}(K)_e$$

Theorem (known-ish).

If the inner product is either $\langle \cdot, \cdot \rangle_{\mathrm{tr}}$ or $\langle \cdot, \cdot \rangle_L$, then

$$\operatorname{JAut}(V) = \operatorname{Aut}(K)_e = \operatorname{Aut}(K) \cap \operatorname{Isom}(V)$$

Simple EJAs

Proposition (Faraut and Korányi, III.4.1–2).

If $V \neq \{0\}$ is a simple EJA, there exists a unique $\alpha > 0$ such that $\langle \cdot, \cdot \rangle = \alpha \langle \cdot, \cdot \rangle_{\text{tr}}$.

In particular,

$$\langle \cdot, \cdot \rangle_L = \frac{\dim(V)}{\operatorname{rank}(V)} \langle \cdot, \cdot \rangle_{\text{tr}}$$

Theorem (Chua 2008).

If the inner product is $\langle \cdot, \cdot \rangle_L$, then

$$\operatorname{JAut}(V) = \operatorname{Aut}(K)_e = \operatorname{Aut}(K) \cap \operatorname{Isom}(V)$$

The proof of this is a few lines of calculus.

Note.

Chua also gives a proof for $\langle \cdot, \cdot \rangle_{\mathrm{tr}}$ but the paragraph he cites in F&K contains some imperfections.

Corollary (cf. Gowda 2017).

If V is a simple EJA with cone of squares K and inner product $\langle \cdot, \cdot \rangle_{\mathrm{tr}}$, then

$$\mathrm{JAut}\left(V\right) = \mathrm{Aut}\left(K\right)_{1_V} = \mathrm{Aut}\left(K\right) \cap \mathrm{Isom}\left(V\right)$$

Proof.

V is simple so $\langle \cdot, \cdot \rangle_{\mathrm{tr}} = \frac{1}{\alpha} \langle \cdot, \cdot \rangle_{L}$. The factor $\frac{1}{\alpha}$ doesn't change the unit e or any of the groups involved, so we cite the theorem for $\langle \cdot, \cdot \rangle_{L}$. □

Non-simple EJAs

Counterexample.

Let $V \coloneqq \mathbb{R} \times \mathbb{R}$ with componentwise multiplication and inner product

$$\langle (x_1, x_2), (y_1, y_2) \rangle \coloneqq x_1 y_1 + 2 x_2 y_2$$

Define $\phi((x_1, x_2)) \coloneqq (x_2, x_1)$. Then $\phi \in \operatorname{JAut}(V)$ but ϕ does not preserve the norm of $x \coloneqq (1, 0)$.

All counterexamples turn out to be of this form.

Question.

What went wrong?

Answer.

- First copy of $\mathbb{R}$ has $\langle \cdot, \cdot \rangle = 1 \langle \cdot, \cdot \rangle_{\text{tr}}$
- Second copy of $\mathbb{R}$ has $\langle \cdot, \cdot \rangle = 2 \langle \cdot, \cdot \rangle_{\text{tr}}$
- Jordan automorphisms ignore $\langle \cdot, \cdot \rangle$
- So, swapping the two factors is an automorphism but not an isometry

Lemma.

Jordan isomorphisms preserve $\langle \cdot, \cdot \rangle_{\mathrm{tr}}$.

Warning.

The meaning of $\langle \cdot, \cdot \rangle_{\mathrm{tr}}$ depends on the EJA.

Proof.

Jordan isomorphisms preserve multiplication, thus characteristic polynomials, thus eigenvalues, thus tr, thus $\langle \cdot, \cdot \rangle_{\mathrm{tr}}$. □

Proposition (Orlitzky 2025).

Let V be an EJA with $\langle \cdot, \cdot \rangle$ given, and U_1, U_2 be nontrivial simple Jordan-isomorphic subalgebras.

Suppose the restrictions of $\langle \cdot, \cdot \rangle$ to U_1 and U_2 are $\alpha_1 \langle \cdot, \cdot \rangle_{\text{tr}}$ and $\alpha_2 \langle \cdot, \cdot \rangle_{\text{tr}}$, respectively.

Then $\alpha_1 = \alpha_2$ if and only if every Jordan isomorphism $\psi : U_1 \to U_2$ is an isometry with respect to $\langle \cdot, \cdot \rangle$.

Proof (first direction).

If $\alpha_1 = \alpha_2$ and if $x, y \in U_1$, then

$$\langle x, y \rangle = \alpha_1 \langle x, y \rangle_{\text{tr}} = \alpha_2 \langle x, y \rangle_{\text{tr}}$$

But Jordan isomorphisms preserve $\langle \cdot, \cdot \rangle_{\text{tr}}$, so for any Jordan isomorphism $\psi : U_1 \to U_2$,

$$\begin{aligned} \langle x, y \rangle &= \alpha_1 \langle x, y \rangle_{\text{tr}} \\ &= \alpha_2 \langle x, y \rangle_{\text{tr}} \\ &= \alpha_2 \langle \psi(x), \psi(y) \rangle_{\text{tr}} \\ &= \langle \psi(x), \psi(y) \rangle \end{aligned}$$

□

Proof (second direction).

If $\psi : U_1 \to U_2$ is a Jordan isomorphism, then by assumption, it's an isometry. So for $x, y \in U_1$,

$$\begin{aligned}
\alpha_1 \langle x, y \rangle_{\text{tr}} &= \langle x, y \rangle \\
&= \langle \psi(x), \psi(y) \rangle \\
&= \alpha_2 \langle \psi(x), \psi(y) \rangle_{\text{tr}} \\
&= \alpha_2 \langle x, y \rangle_{\text{tr}}
\end{aligned}$$

implying that $\alpha_1 = \alpha_2$. □

Simplification.

Pretend that every symmetric cone is of the form

$$K = K_1 \times J \times J$$

in the Euclidean Jordan algebra

$$V = V_1 \times W \times W$$

where V_1, W, W are simple and $V_1 \ncong W$.

Note.

The simple factors of $V = V_1 \times W \times W$ are

$$\begin{aligned} V &= V_1 \times \{0\} \times \{0\} && (1) \\ &+ \{0\} \times W \times \{0\} && (2) \\ &+ \{0\} \times \{0\} \times W && (3) \end{aligned}$$

since the components of V should sum to V.

Theorem (Horne 1978).

$$\mathrm{Aut}\left(K_1 \times J \times J\right)$$

$$=$$

$$\left[\mathrm{Aut}\left(K_1\right) \times \mathrm{Aut}\left(J\right) \times \mathrm{Aut}\left(J\right)\right] \sigma$$

where σ is either the identity map, or

$$\sigma\left(v, w_1, w_2\right) \coloneqq \left(v, w_2, w_1\right)$$

swaps the second and third coordinates.

Theorem (Orlitzky 2025).

$$\operatorname{Aut}(K)_e = \operatorname{Aut}(K) \cap \operatorname{Isom}(V)$$

if and only if every Jordan isomorphism between simple factors of V is an isometry.

Corollary.

$$\operatorname{JAut}(V) = \operatorname{Aut}(K)_e = \operatorname{Aut}(K) \cap \operatorname{Isom}(V)$$

under (only) those same circumstances.

Proof (easy direction).

If $\psi : U_1 \to U_2$ is a Jordan isomorphism but not an isometry, then without loss of generality[†],

$$U_1 = \{0\} \times W \times \{0\}$$
$$U_2 = \{0\} \times \{0\} \times W$$

Now $\phi \coloneqq \operatorname{id} \times \psi \times \psi^{-1} \in \operatorname{JAut}(V) = \operatorname{Aut}(K)_e$ but $\phi \notin \operatorname{Isom}(V)$, so

$$\operatorname{Aut}(K)_e \neq \operatorname{Aut}(K) \cap \operatorname{Isom}(V) \qquad \square$$

† The result for simple EJAs precludes $U_1 = U_2$

Proof (hard direction, pt.1).

Let $A \in \operatorname{Aut}(K)$. We claim that $A(e) \neq e$ implies $A \notin \operatorname{Isom}(V)$. From the Horne decomposition,

$$A = [A_1 \times A_2 \times A_3]\,\sigma$$

If $e = (e_1, e_2, e_3)$, then $A(e) \neq e$ implies that $A_i(e_i) \neq e_i$ for some i. The result for simple EJAs says that A_i is not an isometry. Thus, A is not an isometry:

$$\operatorname{Aut}(K) \cap \operatorname{Isom}(V) \subseteq \operatorname{Aut}(K)_e \qquad \square$$

Proof (hard direction, pt.2).

Let $A \in \operatorname{Aut}(K)_e$. From the Horne decomposition,

$$A = \left[A_1 \times A_2 \times A_3\right] \sigma$$

where each A_i fixes e_i. By the result for simple EJAs, each A_i is an isometry. The permutation σ is a Jordan isomorphism between simple components and by assumption is an isometry. Thus A is an isometry:

$$\operatorname{Aut}(K)_e \subseteq \operatorname{Aut}(K) \cap \operatorname{Isom}(V) \qquad \square$$

Theorem (just proved).

$\text{Aut}(K)_e = \text{Aut}(K) \cap \text{Isom}(V)$ if and only if every Jordan isomorphism between simple factors of V is an isometry. And in any case, $\text{JAut}(V) = \text{Aut}(K)_e$.

Corollary (cf. Faraut and Korányi, p.57).

Under either $\langle \cdot, \cdot \rangle_{\text{tr}}$ or $\langle \cdot, \cdot \rangle_L$,

$$\text{JAut}(V) = \text{Aut}(K)_e = \text{Aut}(K) \cap \text{Isom}(V)$$

Question.

Isn't $K \coloneqq K_1 \times J \times J$ an over-simplification?

Answer.

Indeed! Up to isomorphism, our simplification is reasonable…but the point of the theorem is that some isomorphisms void the result.

The paper† does address the general case.

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† `https://optimization-online.org/2024/11/jordan-and-isometric-cone-automorphisms-in-euclidean-jordan-algebras/`

The end