Jordan automorphisms and derivatives of symmetric cones

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Section 1

Motivation

Motivation

Theorem (Ito/Lourenço 2023).

$$\operatorname{Aut}\left(K_{p,e}^{(i)}\right) = \operatorname{Aut}\left(K_{p,e}\right) \cap \operatorname{Aut}\left(\mathbb{R}_{+} e\right)$$

where

- $K_{p,e}$ is a hyperbolicity cone
- $K_{p,e}^{(i)}$ is its ith Renegar derivative
- some technical conditions have been omitted

Motivation

Observation.

If G_e denotes a stabilizer subgroup of G, then

$$\operatorname{Aut}\left(K_{p,e}\right) \cap \operatorname{Aut}\left(\mathbb{R}_{+} e\right) = \mathbb{R}_{++} \operatorname{Aut}\left(K_{p,e}\right)_{e}$$

and it follows that

$$\operatorname{Aut}\left(K_{p,e}^{(i)}\right) = \mathbb{R}_{++} \operatorname{Aut}\left(K_{p,e}\right)_{e}$$

Motivation

Lemma (Gowda, 2017). If K is the cone of squares in a simple Euclidean Jordan algebra V and if 1_V is its unit element,

$$\operatorname{Aut}(K)_{1_V} = \operatorname{JAut}(V)$$

Recall:

$$\operatorname{Aut}\left(K_{p,e}^{(i)}\right) = \mathbb{R}_{++} \operatorname{Aut}\left(K_{p,e}\right)_{e}$$

Motivation

From this we are inspired to

1. Extend Gowda's result to a non-simple EJA
2. Make $K_{p,e}$ be the cone of squares
3. Paste the previous two results together:

$$\operatorname{Aut}\left(K_{p,e}^{(i)}\right) = \mathbb{R}_{++} \operatorname{JAut}(V)$$

4. Find $\operatorname{JAut}(V)$ where possible

EJA Introduction

EJA Introduction

An EJA (call it V) is an algebra:

- it's a vector space over $\mathbb{R}$
- it's finite-dimensional
- it has a commutative bilinear multiplication
- with a unit element 1_V
- and cone of squares $K = \{x^2 \mid x \in V\}$
- the cone of squares is symmetric

EJA Introduction

- every $x \in V$ has a spectral decomposition,

$$x = \lambda_1(x)\,c_1 + \cdots + \lambda_r(x)\,c_r$$

- the cone of squares K is the set of elements x having all $\lambda_i(x) \ge 0$
- there's a determinant, $\det(x) \coloneqq \prod_{i=1}^{r} \lambda_i(x)$
- $\operatorname{Aut}(K)$ denotes linear automorphisms of K
- $\operatorname{JAut}(V)$ denotes invertible homomorphisms

EJA Introduction

There are “five” simple EJAs,

1. The Jordan spin algebra $\mathcal{L}^n$
2. Real Hermitian matrices $\mathcal{H}^n(\mathbb{R})$
3. Complex Hermitian matrices $\mathcal{H}^n(\mathbb{C})$
4. Quaternion Hermitian matrices $\mathcal{H}^n(\mathbb{H})$
5. Octonion 3×3 Hermitian matrices $\mathcal{H}^3(\mathbb{O})$

EJA Introduction

We pretend all EJAs are of the form

$$V = V_1 \times V_2$$

where

- V_1 and V_2 are simple
- V_1 and V_2 are **not** isomorphic

EJA Introduction

As a result, we pretend that

$$K = K_1 \times K_2$$

is the cone of squares, where

- K_1 and K_2 are symmetric and irreducible
- K_1 and K_2 are **not** isomorphic

EJA Introduction

Theorem (Jordan/von Neumann/Wigner 1934).

This scenario is real life:

- Working up to Jordan isomorphism
- Suppressing repeated factors
- With $N = 2$
- And if we don't care about $V = \{0\}$

(The paper makes no such assumptions.)

Decomposing automorphisms

Decomposing automorphisms

Theorem (Horne 1978).

$$\operatorname{Aut}(K_1 \times K_2) = \operatorname{Aut}(K_1) \times \operatorname{Aut}(K_2)$$

and, consequently,

$$\operatorname{Aut}(K_1 \times K_2)_{(e_1, e_2)}$$

$$=$$

$$\operatorname{Aut}(K_1)_{e_1} \times \operatorname{Aut}(K_2)_{e_2}$$

Decomposing automorphisms

Theorem.

$$\mathrm{JAut}\left(V\right) = \mathrm{Aut}\left(K\right)_{1_V}$$

Proof.

$\mathrm{JAut}\left(V\right)$ is contained in $\mathrm{Aut}\left(K\right)_{1_V}$ because squares and 1_V are preserved when multiplication is.

Decomposing automorphisms

Proof (cont'd).

In the other direction, Horne says that

$$\operatorname{Aut}(K)_{1_V} = \operatorname{Aut}(K_1)_{1_{V_1}} \times \operatorname{Aut}(K_2)_{1_{V_2}}$$

Then from Gowda's simple EJA Lemma,

$$\begin{aligned}\operatorname{Aut}(K)_{1_V} &= \operatorname{JAut}(V_1) \times \operatorname{JAut}(V_2)\\ &\subseteq \operatorname{JAut}(V)\end{aligned}$$

□

Decomposing automorphisms

Remark.

$$\mathrm{JAut}\left(V\right) = \mathrm{Aut}\left(K\right)_{1_V}$$

- Stated without proof by Vinberg in 1965
- Given a proof by Chua 2008
- Appears in 2003 Alfsen and Shultz book

Decomposing automorphisms

Corollary (Gowda/Jeong 2017).

$$\operatorname{JAut}(V_1 \times V_2) = \operatorname{JAut}(V_1) \times \operatorname{JAut}(V_2)$$

Proof. The last line of the preceding proof has

$$\operatorname{Aut}(K)_{1_V} = \operatorname{JAut}(V_1) \times \operatorname{JAut}(V_2)$$

but now $\operatorname{Aut}(K)_{1_V} = \operatorname{JAut}(V)$. □

Section 4

Cone automorphisms

Cone automorphisms

Recall: all EJAs have

$$\operatorname{JAut}(V) = \operatorname{JAut}(V_1) \times \operatorname{JAut}(V_2)$$
$$\operatorname{Aut}(K) = \operatorname{Aut}(K_1) \times \operatorname{Aut}(K_2)$$

and there are only five potential V_i and K_i.

Question. Can we find the five corresponding $\operatorname{JAut}(V_i)$ and $\operatorname{Aut}(K_i)$?

Cone automorphisms

Theorem. If $n \geq 1$, then

$$\operatorname{Aut}\left(\mathcal{L}_{+}^{n}\right) = \left\{ \begin{bmatrix} x_0^2 + \left\|\tilde{x}\right\|^2 & 2x_0\tilde{x}^T U \\ 2x_0\tilde{x} & 2\tilde{x}\tilde{x}^T U + \left(x_0^2 - \left\|\tilde{x}\right\|^2\right) U \end{bmatrix} \right\}$$

where

$$x_0 \in \mathbb{R}$$
$$\tilde{x} \in \mathbb{R}^{n-1}$$
$$0 \leq \left\|\tilde{x}\right\| < x_0$$
$$U \in \operatorname{Isom}\left(\mathbb{R}^{n}\right)$$

Cone automorphisms

The proof is by direct computation of the EJA polar decomposition. The details are unimportant to us.

Remark.

An equivalent description was found a few years ago by Roman Sznajder, but the polar decomposition provides a shortcut.

Cone automorphisms

Proposition. In $\mathcal{H}^n(\mathbb{H})$ the cone of squares is the quaternion PSD cone.

Proof.

Same as over $\mathbb{R}$ or $\mathbb{C}$ using Rodman's *Topics in Quaternion Linear Algebra* for the spectral theory: diagonalize to UDU^*, take $\sqrt{D}$ which has nonnegative entries, etc. □

Cone automorphisms

Theorem.

$$\begin{aligned}\operatorname{Aut}\left(\mathcal{H}_{+}^{n}\left(\mathbb{R}\right)\right) &= \left\{X \mapsto U^{*}XU \mid U \in \mathrm{GL}_{n}\left(\mathbb{R}\right)\right\}\\ \operatorname{Aut}\left(\mathcal{H}_{+}^{n}\left(\mathbb{C}\right)\right) &= \left\{X \mapsto U^{*}XU \mid U \in \mathrm{GL}_{n}\left(\mathbb{C}\right)\right\}\\ &\quad \cup \left\{X \mapsto U^{*}\overline{X}U \;\middle|\; U \in \mathrm{GL}_{n}\left(\mathbb{C}\right)\right\}\\ \operatorname{Aut}\left(\mathcal{H}_{+}^{n}\left(\mathbb{H}\right)\right) &= \left\{X \mapsto U^{*}XU \mid U \in \mathrm{GL}_{n}\left(\mathbb{H}\right)\right\}\end{aligned}$$

Proof. Direct consequence of Schneider/Rodman inertia theorems over $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$. $\square$

Jordan automorphisms

Jordan automorphisms

Theorem.

$$
\begin{aligned}
\mathrm{JAut}\left(\mathcal{L}^n\right) &= \left\{\mathrm{id}_{\mathbb{R}} \times U \,\middle|\, U \in \mathrm{Isom}\left(\mathbb{R}^{n-1}\right)\right\} \\
\mathrm{JAut}\left(\mathcal{H}^n\left(\mathbb{R}\right)\right) &= \left\{X \mapsto U^* X U \mid U \in \mathrm{Isom}\left(\mathbb{R}^n\right)\right\} \\
\mathrm{JAut}\left(\mathcal{H}^n\left(\mathbb{C}\right)\right) &= \left\{X \mapsto U^* X U \mid U \in \mathrm{Isom}\left(\mathbb{C}^n\right)\right\} \\
&\cup \left\{X \mapsto U^* \overline{X} U \,\middle|\, U \in \mathrm{Isom}\left(\mathbb{C}^n\right)\right\} \\
\mathrm{JAut}\left(\mathcal{H}^n\left(\mathbb{H}\right)\right) &= \left\{X \mapsto U^* X U \mid U \in \mathrm{Isom}\left(\mathbb{H}^n\right)\right\} \\
\mathrm{JAut}\left(\mathcal{H}^3\left(\mathbb{O}\right)\right) &= \text{the exceptional Lie group } F_4
\end{aligned}
$$

Proof.

Gowda, Tao, and Sznajder found $\operatorname{JAut}(\mathcal{L}^n)$ and $\operatorname{JAut}(\mathcal{H}^n(\mathbb{R}))$ in 2004. Chevalley and Shafer found $\operatorname{JAut}(\mathcal{H}^3(\mathbb{O}))$ in 1950.

For the others, use $\operatorname{JAut}(V) = \operatorname{Aut}(K)_{1_V}$ with $1_V = I$ and the characterization of $\operatorname{Aut}(K)$. □

Remark.

If the automorphisms of $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ are known, a 2008 theorem of Huang can be used for $\operatorname{JAut}(\mathcal{H}^n(\mathbb{A}))$ when $n \geq 3$.

Rodman's book contains the automorphisms of $\mathbb{H}$, and we get the same result either way.

Jordan automorphisms

Remark.

A 1947 result of Kalisch gives an isomorphic representation of $\operatorname{JAut}\left(\mathcal{H}^n\left(\mathbb{A}\right)\right)$ for $\mathbb{A} \in \left\{\mathbb{R}, \mathbb{H}\right\}$.

Jordan automorphisms

Remark.

In $\operatorname{JAut}\left(\mathcal{H}^n\left(\mathbb{C}\right)\right)$, the maps

$$\left\{X \mapsto U^* \overline{X} U \,\middle|\, U \in \operatorname{Isom}\left(\mathbb{C}^n\right)\right\}$$

are **not** redundant. They cannot be written as $X \mapsto V^* X V$ for $V \in \operatorname{Isom}\left(\mathbb{C}^n\right)$, without the conjugation.

Proposition.

The right-eigenvalues of a matrix in $\mathcal{H}^n(\mathbb{H})$ are the same as its Jordan-algebraic eigenvalues.

Proof.

EJA/matrix diagonalization produces EJA/matrix eigenvalues. The form of $\mathrm{JAut}(\mathcal{H}^n(\mathbb{H}))$ shows that they're the same process. $\square$

Jordan automorphisms

Theorem.

1. $\mathrm{JAut}\left(\mathcal{L}^n\right)$ is path-connected if $n \in \{0,1\}$ and disconnected otherwise
2. $\mathrm{JAut}\left(\mathcal{H}^n\left(\mathbb{R}\right)\right)$ is path-connected if n is odd and disconnected otherwise
3. $\mathrm{JAut}\left(\mathcal{H}^n\left(\mathbb{C}\right)\right)$ is disconnected
4. $\mathrm{JAut}\left(\mathcal{H}^n\left(\mathbb{H}\right)\right)$ is path-connected
5. $\mathrm{JAut}\left(\mathcal{H}^3\left(\mathbb{O}\right)\right)$ is path-connected

Jordan automorphisms

Proof.

Two are easy:

- $\mathrm{JAut}\left(\mathcal{L}^n\right) \cong \mathrm{Isom}\left(\mathbb{R}^{n-1}\right)$ has two components for $n \geq 2$
- $\mathrm{JAut}\left(\mathcal{H}^3\left(\mathbb{O}\right)\right)$ is simply connected (Yokota's book)

Jordan automorphisms

Proof (cont'd).

$\mathrm{JAut}\left(\mathcal{H}^n\left(\mathbb{H}\right)\right)$ is also not bad:

- $\mathrm{Isom}\left(\mathbb{H}^n\right)$ is path-connected (Tapp's book). Define,

  $$\varphi_U \coloneqq X \mapsto U^* X U$$

  The path from U to V in $\mathrm{Isom}\left(\mathbb{H}^n\right)$ induces a path from φ_U to φ_V in $\mathrm{JAut}\left(\mathcal{H}^n\left(\mathbb{H}\right)\right)$.

Jordan automorphisms

Proof (cont'd).

In $\text{JAut}(\mathcal{H}^n(\mathbb{C}))$ we saw that $\varphi_U \coloneqq X \mapsto U^* X U$ and $\psi_V \coloneqq X \mapsto V^* \overline{X} V$ cannot be equal.

1. $\text{JAut}(\mathcal{H}^n(\mathbb{C}))$ is a disjoint union...
2. ...of continuous images of $\text{Isom}(\mathbb{C}^n)$
3. Closed disjoint sets are separated

Jordan automorphisms

Proof (cont'd). For $\mathrm{JAut}\left(\mathcal{H}^n\left(\mathbb{R}\right)\right)$:

- EJAs are real vector spaces
- $\mathrm{JAut}\left(\mathcal{H}^n\left(\mathbb{R}\right)\right)$ preserves the trace norm
- So $\mathrm{JAut}\left(\mathcal{H}^n\left(\mathbb{R}\right)\right) \cong \mathrm{Isom}\left(\mathbb{R}^k\right)$ for some k
- A priori, two components
- $\det\left(-\mathrm{id}_{\mathbb{R}} \times I\right) = -1$ for even n
- otherwise $\varphi_U = \varphi_{-U}$ and $\det\left(-U\right) = \det\left(V\right)$ lets you make a path between φ_U, φ_V $\square$

Section 6

Hyperbolicity cones

Hyperbolicity cones

Definition. The polynomial

$$p \in \mathbb{R}\left[X_1, X_2, \ldots, X_n\right]$$

is hyperbolic along $e \in \mathbb{R}^n$ if,

- p is homogeneous
- $p\left(e\right) > 0$
- all roots of $\lambda \mapsto p\left(\lambda e - x\right)$ are real

Hyperbolicity cones

The roots of $\lambda \mapsto p(\lambda e - x)$ are called the eigenvalues of x.

The *hyperbolicity cone* of p along e is

$$K_{p,e} \coloneqq \{x \in \mathbb{R}^n \mid p(\lambda e - x) \neq 0 \text{ for all } \lambda < 0\}$$

and is the set where all eigenvalues are nonnegative.

Hyperbolicity cones

Example.

In a Euclidean Jordan algebra:

- The determinant is a homogeneous polynomial
- All eigenvalues are real
- The determinant is hyperbolic along 1_V
- $K_{\det,1_V}$ is the cone of squares

Hyperbolicity cones

Renegar 2006:

- Take the derivative of p along e
- Get a new hyperbolicity cone $K_{p,e}^{(1)}$
- $K_{p,e}^{(1)}$ is a relaxation of $K_{p,e}$
- Repeat:

$$K_{p,e} \subseteq K_{p,e}^{(1)} \subseteq \cdots \subseteq K_{p,e}^{(i)}$$

Hyperbolicity cones

Recall:

Theorem (Ito/Lourenço 2023).

Subject to some technical conditions,

$$\operatorname{Aut}\left(K_{p,e}^{(i)}\right) = \mathbb{R}_{++} \operatorname{Aut}\left(K_{p,e}\right)_e$$

Theorem. Let V be an EJA of rank $r \geq 4$ and $1 \leq i \leq r - 3$. Then in coordinates,

$$\operatorname{Aut}\left(K^{(i)}_{\det,1_V}\right) = \mathbb{R}_{++}\operatorname{JAut}(V)$$

Proof. Substitute into the Ito/Lourenço result:

$$p = \det$$

$$e = 1_V$$

$$\operatorname{JAut}(V) = \operatorname{Aut}(K)_{1_V}$$

□

Section 7

Summary

Summary

- New proof of $\mathrm{JAut}(V) = \mathrm{Aut}(K)_{1_V}$
- New proof of

$$\mathrm{JAut}(V_1 \times V_2) = \mathrm{JAut}(V_1) \times \mathrm{JAut}(V_2)$$

- New description of $\mathrm{Aut}(\mathcal{L}^n_+)$
- Found $\mathrm{Aut}(\mathcal{H}^n_+(\mathbb{H}))$
- Found $\mathrm{JAut}(\mathcal{H}^n(\mathbb{C}))$ and $\mathrm{JAut}(\mathcal{H}^n(\mathbb{H}))$
- Path-connectedness of $\mathrm{JAut}(V)$
- Found $\mathrm{Aut}\left(K^{(i)}_{\det,1_V}\right)$ in an EJA

Section 8

The end