Jordan automorphisms and derivatives of symmetric cones

Michael Orlitzky

ILAS Madrid, Tuesday, June 13th, 2023
Motivation
Motivation

Theorem (Ito/Lourenço 2023).

\[
\text{Aut} \left( K^{(i)}_{p,e} \right) = \text{Aut} \left( K_{p,e} \right) \cap \text{Aut} \left( \mathbb{R}_+ e \right)
\]

where

- \( K_{p,e} \) is a hyperbolicity cone
- \( K^{(i)}_{p,e} \) is its \( i \)th Renegar derivative
- some technical conditions have been omitted
Observation.

If $G_e$ denotes a stabilizer subgroup of $G$, then

$$\text{Aut}(K_{p,e}) \cap \text{Aut}(\mathbb{R}+e) = \mathbb{R}++ \text{Aut}(K_{p,e})_e$$

and it follows that

$$\text{Aut}(K_{p,e}^{(i)}) = \mathbb{R}++ \text{Aut}(K_{p,e})_e$$
Lemma (Gowda, 2017). If $K$ is the cone of squares in a simple Euclidean Jordan algebra $V$ and if $1_V$ is its unit element,

$$\text{Aut}(K)_{1_V} = J\text{Aut}(V)$$

Recall:

$$\text{Aut}\left(K_{p,e}^{(i)}\right) = \mathbb{R}_{++} \text{Aut}\left(K_{p,e}\right)_e$$
Motivation

From this we are inspired to

1. Extend Gowda’s result to a non-simple EJA
2. Make $K_{p,e}$ be the cone of squares
3. Paste the previous two results together:

$$\text{Aut} \left( K_{p,e}^{(i)} \right) = \mathbb{R}_{++} \text{JAut} \left( V \right)$$

4. Find JAut $(V)$ where possible
Section 2

EJA Introduction
EJA Introduction

An EJA (call it $V$) is an algebra:

- it’s a vector space over $\mathbb{R}$
- it’s finite-dimensional
- it has a commutative bilinear multiplication
- with a unit element $1_V$
- and cone of squares $K = \{x^2 \mid x \in V\}$
- the cone of squares is symmetric
EJA Introduction

- every \( x \in V \) has a spectral decomposition,

\[
x = \lambda_1(x)c_1 + \cdots + \lambda_r(x)c_r
\]

- the cone of squares \( K \) is the set of elements \( x \) having all \( \lambda_i(x) \geq 0 \)

- there's a determinant, \( \det(x) := \prod_{i=1}^{r} \lambda_i(x) \)

- \( \text{Aut}(K) \) denotes linear automorphisms of \( K \)

- \( \text{JAut}(V) \) denotes invertible homomorphisms
There are “five” simple EJAs,

1. The Jordan spin algebra $\mathcal{L}^n$
2. Real Hermitian matrices $\mathcal{H}^n (\mathbb{R})$
3. Complex Hermitian matrices $\mathcal{H}^n (\mathbb{C})$
4. Quaternion Hermitian matrices $\mathcal{H}^n (\mathbb{H})$
5. Octonion 3 × 3 Hermitian matrices $\mathcal{H}^3 (\mathbb{O})$
EJA Introduction

We pretend all EJAs are of the form

\[ V = V_1 \times V_2 \]

where

- \( V_1 \) and \( V_2 \) are simple
- \( V_1 \) and \( V_2 \) are not isomorphic
As a result, we pretend that

\[ K = K_1 \times K_2 \]

is the cone of squares, where

- \( K_1 \) and \( K_2 \) are symmetric and irreducible
- \( K_1 \) and \( K_2 \) are not isomorphic
Theorem (Jordan/von Neumann/Wigner 1934).

This scenario is real life:

- Working up to Jordan isomorphism
- Suppressing repeated factors
- With $N = 2$
- And if we don’t care about $V = \{0\}$

(The paper makes no such assumptions.)
Decomposing automorphisms
Decomposing automorphisms

Theorem (Horne 1978).

\[
\text{Aut} \left(K_1 \times K_2\right) = \text{Aut} \left(K_1\right) \times \text{Aut} \left(K_2\right)
\]

and, consequently,

\[
\text{Aut} \left(K_1 \times K_2\right)_{(e_1,e_2)} = \text{Aut} \left(K_1\right)_{e_1} \times \text{Aut} \left(K_2\right)_{e_2}
\]
Theorem.

$$\text{JAut} \ (V) = \text{Aut} \ (K)_{1_V}$$

Proof.

$\text{JAut} \ (V)$ is contained in $\text{Aut} \ (K)_{1_V}$ because squares and $1_V$ are preserved when multiplication is.
Proof (cont’d).

In the other direction, Horne says that

\[ \text{Aut} (K)_1 = \text{Aut} (K_1)_1 \times \text{Aut} (K_2)_1 \]

Then from Gowda’s simple EJA Lemma,

\[ \text{Aut} (K)_1 = J\text{Aut} (V_1) \times J\text{Aut} (V_2) \subseteq J\text{Aut} (V) \]
Decomposing automorphisms

Remark.

\[ J\text{Aut} (V) = \text{Aut} (K)_{1V} \]

• Stated without proof by Vinberg in 1965
• Given a proof by Chua 2008
• Appears in 2003 Alfsen and Shultz book
**Decomposing Automorphisms**

**Corollary (Gowda/Jeong 2017).**

\[ \text{JAut} (V_1 \times V_2) = \text{JAut} (V_1) \times \text{JAut} (V_2) \]

**Proof.** The last line of the preceding proof has

\[ \text{Aut} (K)_{1V} = \text{JAut} (V_1) \times \text{JAut} (V_2) \]

but now \( \text{Aut} (K)_{1V} = \text{JAut} (V) \). \( \square \)
Section 4

Cone automorphisms
Recall: all EJAs have

\[ \text{JAut}(V) = \text{JAut}(V_1) \times \text{JAut}(V_2) \]
\[ \text{Aut}(K) = \text{Aut}(K_1) \times \text{Aut}(K_2) \]

and there are only five potential \( V_i \) and \( K_i \).

**Question.** Can we find the five corresponding \( \text{JAut}(V_i) \) and \( \text{Aut}(K_i) \)?
**Theorem.** If $n \geq 1$, then

$$\text{Aut} ( \mathcal{L}_+^n ) = \left\{ \begin{bmatrix} x_0^2 + \| \tilde{x} \|^2 & 2x_0 \tilde{x}^T U \\ 2x_0 \tilde{x} & 2\tilde{x} \tilde{x}^T U + (x_0^2 - \| \tilde{x} \|^2) U \end{bmatrix} \right\}$$

where

- $x_0 \in \mathbb{R}$
- $\tilde{x} \in \mathbb{R}^{n-1}$
- $0 \leq \| \tilde{x} \| < x_0$
- $U \in \text{Isom} (\mathbb{R}^n)$
Cone Automorphisms

The proof is by direct computation of the EJA polar decomposition. The details are unimportant to us.

Remark.

An equivalent description was found a few years ago by Roman Sznajder, but the polar decomposition provides a shortcut.
Cone automorphisms

Proposition. In $\mathcal{H}^n (\mathbb{H})$ the cone of squares is the quaternion PSD cone.

Proof.

Same as over $\mathbb{R}$ or $\mathbb{C}$ using Rodman’s *Topics in Quaternion Linear Algebra* for the spectral theory: diagonalize to $UDU^*$, take $\sqrt{D}$ which has nonnegative entries, etc.
Cone automorphisms

Theorem.

\[
\begin{align*}
\text{Aut} \left( \mathcal{H}^n_+ (\mathbb{R}) \right) &= \{ X \mapsto U^* X U \mid U \in \text{GL}_n (\mathbb{R}) \} \\
\text{Aut} \left( \mathcal{H}^n_+ (\mathbb{C}) \right) &= \{ X \mapsto U^* X U \mid U \in \text{GL}_n (\mathbb{C}) \} \\
&\quad \cup \{ X \mapsto U^* \overline{X} U \mid U \in \text{GL}_n (\mathbb{C}) \} \\
\text{Aut} \left( \mathcal{H}^n_+ (\mathbb{H}) \right) &= \{ X \mapsto U^* X U \mid U \in \text{GL}_n (\mathbb{H}) \}
\end{align*}
\]

Proof. Direct consequence of Schneider/Rodman inertia theorems over \( \mathbb{R}, \mathbb{C}, \) and \( \mathbb{H} \).
Section 5

Jordan automorphisms
Theorem.

\[
\begin{align*}
\text{JAut} (\mathcal{L}^n) &= \{ \text{id}_\mathbb{R} \times U \mid U \in \text{Isom} (\mathbb{R}^{n-1}) \} \\
\text{JAut} (\mathcal{H}^n (\mathbb{R})) &= \{ X \mapsto U^* X U \mid U \in \text{Isom} (\mathbb{R}^n) \} \\
\text{JAut} (\mathcal{H}^n (\mathbb{C})) &= \{ X \mapsto U^* X U \mid U \in \text{Isom} (\mathbb{C}^n) \} \\
&\quad \cup \{ X \mapsto U^* \overline{X} U \mid U \in \text{Isom} (\mathbb{C}^n) \} \\
\text{JAut} (\mathcal{H}^n (\mathbb{H})) &= \{ X \mapsto U^* X U \mid U \in \text{Isom} (\mathbb{H}^n) \} \\
\text{JAut} (\mathcal{H}^3 (\mathbb{O})) &= \text{the exceptional Lie group } F_4
\end{align*}
\]
Proof.

Gowda, Tao, and Sznajder found $J_{\text{Aut}}(\mathcal{L}^n)$ and $J_{\text{Aut}}(\mathcal{H}^n(\mathbb{R}))$ in 2004. Chevalley and Shafer found $J_{\text{Aut}}(\mathcal{H}^3(\mathbb{O}))$ in 1950.

For the others, use $J_{\text{Aut}}(V) = \text{Aut}(K)_{1_V}$ with $1_V = I$ and the characterization of Aut$(K)$. □
Remark.

If the automorphisms of $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ are known, a 2008 theorem of Huang can be used for $\text{JAut}(\mathcal{H}^n(\mathbb{A}))$ when $n \geq 3$.

Rodman's book contains the automorphisms of $\mathbb{H}$, and we get the same result either way.
Remark.

A 1947 result of Kalisch gives an isomorphic representation of $\text{JAut} (\mathcal{H}^n (\mathbb{A}))$ for $\mathbb{A} \in \{ \mathbb{R}, \mathbb{H} \}$. 
Remark.

In $\mathrm{JAut} (\mathcal{H}^n (\mathbb{C}))$, the maps

$$\{ X \mapsto U^* \overline{X} U \mid U \in \mathrm{Isom} (\mathbb{C}^n) \}$$

are not redundant. They cannot be written as $X \mapsto V^* X V$ for $V \in \mathrm{Isom} (\mathbb{C}^n)$, without the conjugation.
Proposition.

The right-eigenvalues of a matrix in $\mathcal{H}^n (\mathbb{H})$ are the same as its Jordan-algebraic eigenvalues.

Proof.

EJA/matrix diagonalization produces EJA/matrix eigenvalues. The form of JAut (\mathcal{H}^n (\mathbb{H})) shows that they’re the same process.
Theorem.

1. \( \text{JAut}(\mathcal{L}^n) \) is path-connected if \( n \in \{0,1\} \) and disconnected otherwise
2. \( \text{JAut}(\mathcal{H}^n(\mathbb{R})) \) is path-connected if \( n \) is odd and disconnected otherwise
3. \( \text{JAut}(\mathcal{H}^n(\mathbb{C})) \) is disconnected
4. \( \text{JAut}(\mathcal{H}^n(\mathbb{H})) \) is path-connected
5. \( \text{JAut}(\mathcal{H}^3(\mathbb{O})) \) is path-connected
Proof.

Two are easy:

- \( \text{JAut}(\mathcal{L}^n) \cong \text{Isom}(\mathbb{R}^{n-1}) \) has two components for \( n \geq 2 \)
- \( \text{JAut}(\mathcal{H}^3(\mathbb{O})) \) is simply connected (Yokota’s book)
Proof (cont’d).

$\text{JAut} \left( \mathcal{H}^n \left( \mathbb{H} \right) \right)$ is also not bad:

- $\text{Isom} \left( \mathbb{H}^n \right)$ is path-connected (Tapp’s book).

Define,

$$\varphi_U := X \mapsto U^* X U$$

The path from $U$ to $V$ in $\text{Isom} \left( \mathbb{H}^n \right)$ induces a path from $\varphi_U$ to $\varphi_V$ in $\text{JAut} \left( \mathcal{H}^n \left( \mathbb{H} \right) \right)$. 
Proof (cont’d).

In $\text{JAut} (\mathcal{H}^n (\mathbb{C}))$ we saw that $\varphi_U := X \mapsto U^*XU$ and $\psi_V := X \mapsto V^*\overline{X}V$ cannot be equal.

1. $\text{JAut} (\mathcal{H}^n (\mathbb{C}))$ is a disjoint union...
2. ...of continuous images of $\text{Isom} (\mathbb{C}^n)$
3. Closed disjoint sets are separated
Proof (cont’d). For $\text{JAut} \left( \mathcal{H}^n (\mathbb{R}) \right)$:

- EJAs are real vector spaces
- $\text{JAut} \left( \mathcal{H}^n (\mathbb{R}) \right)$ preserves the trace norm
- So $\text{JAut} \left( \mathcal{H}^n (\mathbb{R}) \right) \cong \text{Isom} \left( \mathbb{R}^k \right)$ for some $k$
- A priori, two components
- $\det (\mathbb{id}_{\mathbb{R}^n} \times I) = -1$ for even $n$
- otherwise $\varphi_U = \varphi_{-U}$ and $\det (-U) = \det (V)$ lets you make a path between $\varphi_U, \varphi_V$
Section 6

Hyperbolicity cones
**Definition.** The polynomial

\[ p \in \mathbb{R} [X_1, X_2, \ldots, X_n] \]

is hyperbolic along \( e \in \mathbb{R}^n \) if,

- \( p \) is homogeneous
- \( p(e) > 0 \)
- all roots of \( \lambda \mapsto p(\lambda e - x) \) are real
The roots of $\lambda \mapsto p(\lambda e - x)$ are called the eigenvalues of $x$.

The hyperbolicity cone of $p$ along $e$ is

$$K_{p,e} := \{ x \in \mathbb{R}^n \mid p(\lambda e - x) \neq 0 \text{ for all } \lambda < 0 \}$$

and is the set where all eigenvalues are nonnegative.
Example.

In a Euclidean Jordan algebra:

- The determinant is a homogeneous polynomial
- All eigenvalues are real
- The determinant is hyperbolic along $1_V$
- $K_{\text{det},1_V}$ is the cone of squares
Renegar 2006:

- Take the derivative of $p$ along $e$
- Get a new hyperbolicity cone $K_{p,e}^{(1)}$
- $K_{p,e}^{(1)}$ is a relaxation of $K_{p,e}$
- Repeat:

$$K_{p,e} \subseteq K_{p,e}^{(1)} \subseteq \cdots \subseteq K_{p,e}^{(i)}$$
Recall:

**Theorem (Ito/Lourenço 2023).**

Subject to some technical conditions,

\[ \text{Aut} \left( K_{p,e}^{(i)} \right) = \mathbb{R}_+ \text{Aut} \left( K_{p,e} \right)_e \]
**Theorem.** Let $V$ be an EJA of rank $r \geq 4$ and $1 \leq i \leq r - 3$. Then in coordinates,

$$\text{Aut} \left( K_{\det,1_V}^{(i)} \right) = \mathbb{R}_{++} \text{JAut} (V)$$

**Proof.** Substitute into the Ito/Lourenço result:

$$p = \det$$

$$e = 1_V$$

$$\text{JAut} (V) = \text{Aut} (K)_{1_V}$$
Summary
Summary

- New proof of $\text{JAut}(V) = \text{Aut}(K)_{1^V}$
- New proof of

$$\text{JAut}(V_1 \times V_2) = \text{JAut}(V_1) \times \text{JAut}(V_2)$$

- New description of $\text{Aut}(\mathcal{L}_n^+)$
- Found $\text{Aut}(\mathcal{H}_+^n(\mathbb{H}))$
- Found $\text{JAut}(\mathcal{H}_+^n(\mathbb{C}))$ and $\text{JAut}(\mathcal{H}_+^n(\mathbb{H}))$
- Path-connectedness of $\text{JAut}(V)$
- Found $\text{Aut}(K_{\text{det},1^V}^{(i)})$ in an EJA
Section 8

The end