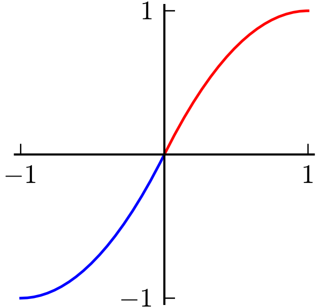
# *Optimal Recovery of Differentiable Functions by Univariate Splines*

Michael Orlitzky

**Definition.** A *spline* is a piecewise-defined polynomial.
**Example.**

$$s(x) = \begin{cases} (x+1)^2 - 1, & x \in [-1, 0] \\ -(1-x)^2 + 1, & x \in [0, 1] \end{cases}$$



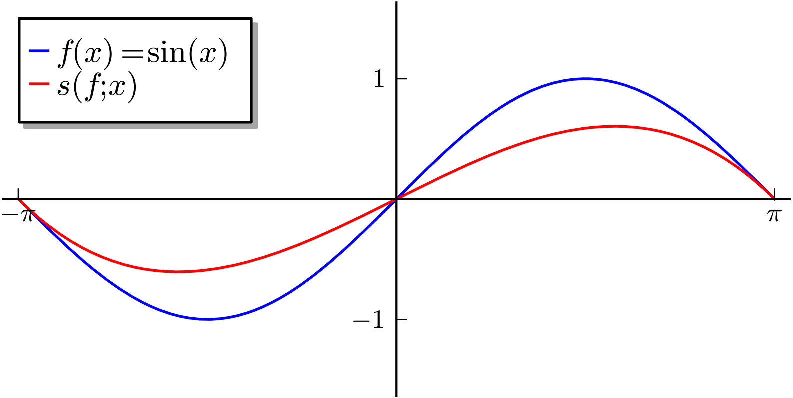
# Polynomial Splines

Splines are used to approximate other functions. Usually, a spline is defined in terms of the values and derivatives of the function it approximates.

**Example.** If we're given the values and first derivatives of a function $f$ at two points, $a$ and $b$, then the spline $s(f;x)$ interpolates $f$ and $f'$ at those points.

$$\begin{aligned}
4 \cdot s(f;x) &= (x^3 - 3x + 2) \cdot f(a) \\
&\quad + (x^3 - x^2 - x + 1) \cdot f'(a) \\
&\quad + (-x^3 + 3x + 2) \cdot f(b) \\
&\quad + (x^3 + x^2 - x - 1) \cdot f'(b)
\end{aligned}$$

If we let $a = -\pi, \; b = \pi$ and substitute $f(x) = \sin(x)$ into this formula, we get a decent approximation of $\sin(x)$ on $[-\pi, \pi]$.



More generally, we can write a spline as,

$$s(f;x) = \sum_{k=0}^{n} A_k(x) \cdot f^{(k)}(a) + B_k(x) \cdot f^{(k)}(b),$$

where $A_k(x)$ and $B_k(x)$ are piecewise polynomials and $f^{(k)}$ denotes the $k^{\text{th}}$ derivative of $f$.

*Note.* This formula doesn't make much sense unless the $k^{\text{th}}$ derivative of $f$ exists. We can formalize this requirement.

# Optimality

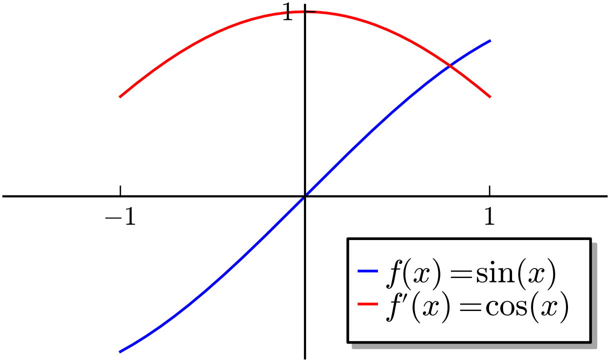
**Definition.** We denote by $W^r$ the space of all functions $f$ defined over $[-1, 1]$ such that $f^{(r-1)}$ is continuous, $f^{(r)}$ is piecewise continuous, and $\|f^{(r)}\|_\infty \leq 1$.

Intuitively, this means that $f$ cannot change too fast.

- Our choice of $[-1, 1]$ here is merely for convenience.
- So is the bound on $\|f^{(r)}\|_\infty$.

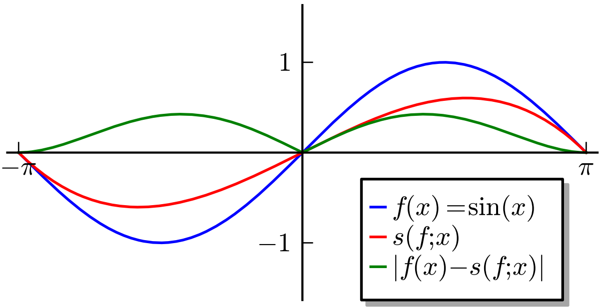
If we restrict ourselves to the class $W^r$, it becomes possible to define an optimal spline.

**Example.** The function $f(x) = \sin(x)$ is in $W^1$ because $\sin(x)$ is continuous, and $f'(x) = \cos(x)$ is continuous and bounded absolutely by 1.



**Definition.** The error of a spline $s(f;x)$ at a point $x$ is $|f(x) - s(f;x)|$; i.e. the difference between the value of the spline and the value of the function it approximates.

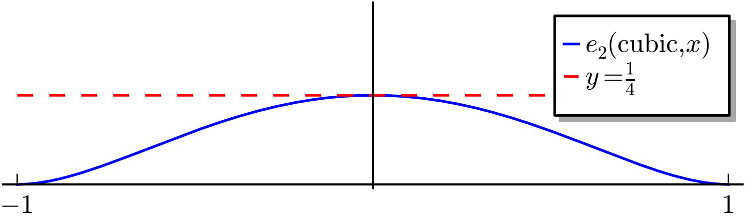
Here, we depict the interpolation by cubic polynomial of $\sin(x)$ along with the error, in green.



**Definition.** The maximal error achieved by the spline $s(f;x)$ at $x$ for any function in $W^r$ is given by,

$$e_r(s;x) = \sup_{f \in W^r} |f(x) - s(f;x)|.$$

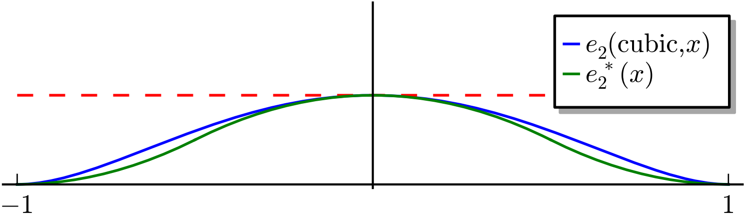
In other words, at each point $x$, there is a function $f$ for which the approximation $s(f;x)$ is worse than for all other functions in $W^r$.



**Definition.** The error at $x$ of the best spline for the worst function $f$ in the class $W^r$ is given by,

$$e_r^*(x) = \inf_s [e_r(s;x)].$$

If we're considering the entire class of functions rather than a *particular* function, $e_r^*(x)$ is the best possible error any spline can achieve at $x$.



# Optimality

**Definition.** We say that a spline $s(f;x)$ is *optimal* on $W^r$ if,

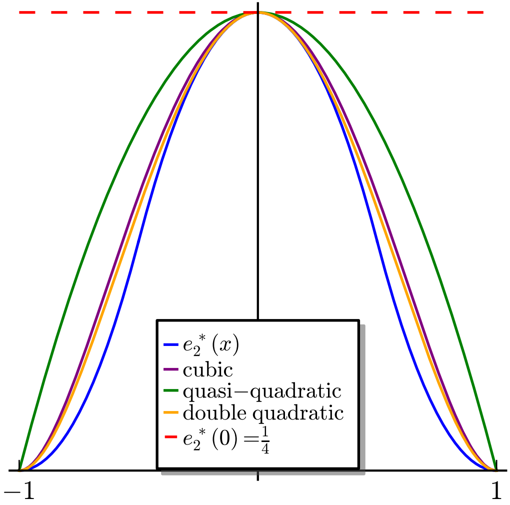
$$e_r(s;x) \leq \sup_{x \in [-1,1]} e_r^*(x), \ x \in [-1,1].$$

In the previous example,

- $\sup_{x \in [-1,1]} e_2^*(x) = e_2^*(0) = \frac{1}{4}$
- $e_2(\text{cubic}, x) \leq \frac{1}{4}, \ x \in [-1,1]$

Therefore, the cubic is optimal on $W^2$.

**Example.** $e_2(s;x)$ for some optimal splines.



# Optimality

**Our goal**: determine whether or not a given spline is optimal.

It turns out, the maximum error of the best spline is always achieved at the midpoint. That is,

$$\sup_{x \in [-1,1]} e_r^*(x) = e_r^*(0).$$

So, the spline $s(f;x)$ is optimal if,

$$e_r(s;x) \leq e_r^*(0).$$

Therefore, we would like to know $e_r^*(0)$.

# Computing $e_r^*(x)$

Boyanov [1] gives us $e_r^*(x)$:

$$e_r^*(x) = \int_{-1}^{x} (x-t)^{r-1} \operatorname{sign}[U_r(t)]\, dt,$$

where $U_r(t)$ is the polynomial of the form $t^r + a_1 t^{r-1} + \cdots + a_r$ that differs least from zero in the interval $[-1, 1]$ with respect to the $L_1$ norm.

However, we would prefer a closed form. To compute $\operatorname{sign}[U_r(t)]$, we need to know the roots of $U_r$.

# COMPUTING $e_r^*(x)$

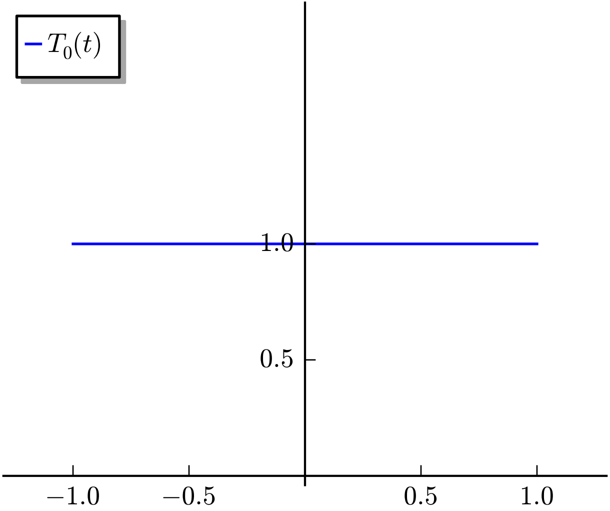
From Powell [2], we know that the polynomial of the form $x^{n+1} + a_1 x^n + \ldots a_m$ differing least from zero in the interval $[-1, 1]$ is,

$$T'_{n+2}(t)/[2^{n+1}(n+2)],$$

where $T_n$ is the $n^{\text{th}}$ Chebychev polynomial.

It can be shown that the $n^{\text{th}}$ Chebychev polynomial is equivalent to $\cos(n \cdot \arccos(t))$, for $n \geq 0$.

**Example.** $T_n$ for small values of $n$.



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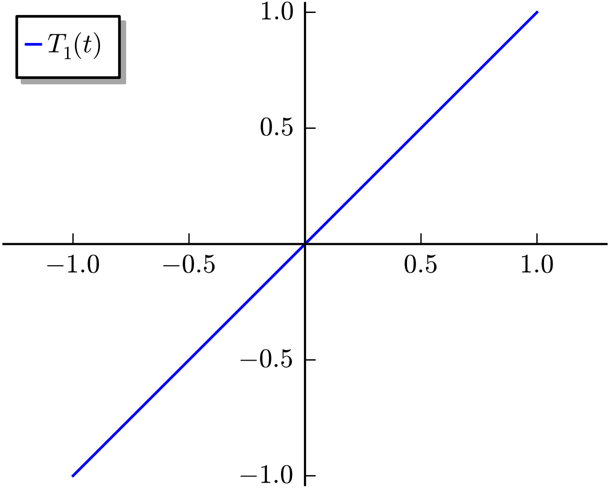

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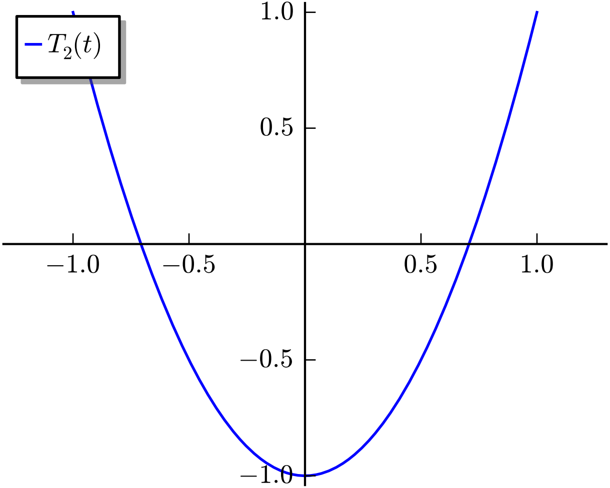

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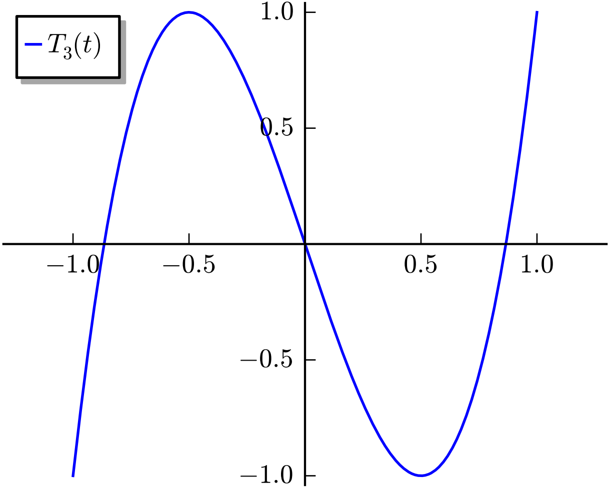
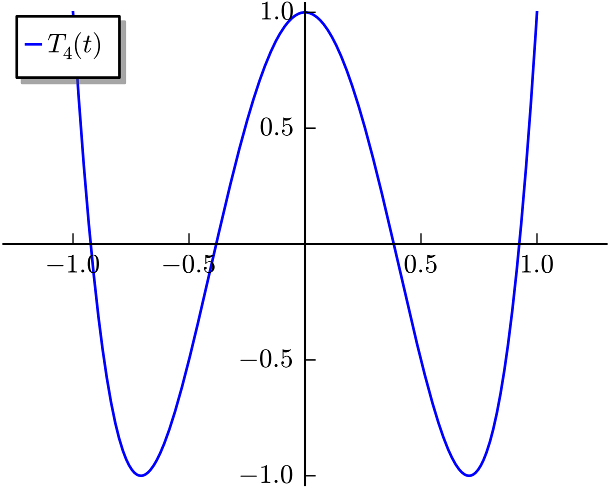


**Example.** $T_n$ for small values of $n$.



Using this formula, we can compute the roots of $U_r$ easily. They are,

$$t = \cos\left(\frac{k\pi}{r+1}\right),\ k = 0 \ldots r+1,$$

and we define,

$$\xi_k = \cos\left(\frac{(r+1-k)\pi}{r+1}\right),\ k = 0 \ldots r+1,$$

so that the roots $\xi_k$ occur in increasing order.

# COMPUTING $e_r^*(x)$

If we evaluate $U_r$ at $t = -1 + \epsilon$, we find that,

$$\text{sign}[U_r(t)] = (-1)^r,\ t \in (\xi_0, \xi_1).$$

By the characterization theorem, $U_r$ must change sign at every root. We know the roots, and therefore, we know $\text{sign}[U_r(t)]$ on all of $[-1, 1]$!

$$\text{sign}[U_r(t)] = (-1)^{r+i},\ t \in (\xi_i, \xi_{i+1}).$$

# COMPUTING $e_r^*(x)$

Now we just integrate.

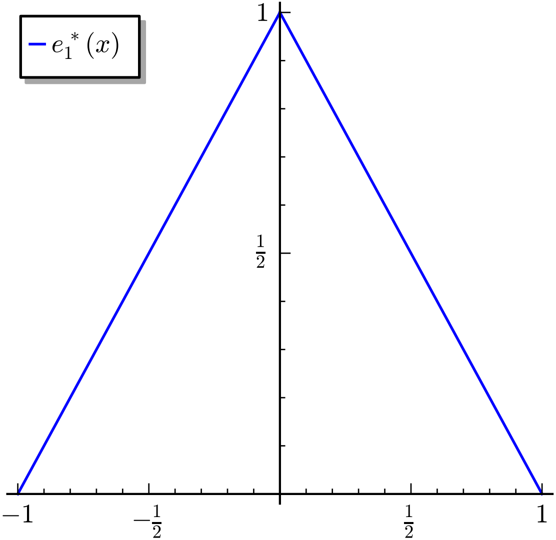
For $x \in [-1, \xi_1]$,

$$r! \cdot e_r^*(x) = (-1 - x)^r$$

And for $x \in [\xi_i, \xi_{i+1}]$,

$$\begin{aligned} r! \cdot e_r^*(x) &= \sum_{k=0}^{i-1} (-1)^{r+k-1} \left[ (x - \xi_{k+1})^r - (x - \xi_k)^r \right] \\ &\quad + (-1)^i (\xi_i - x)^r \end{aligned}$$

# Computing $e_r^*(x)$



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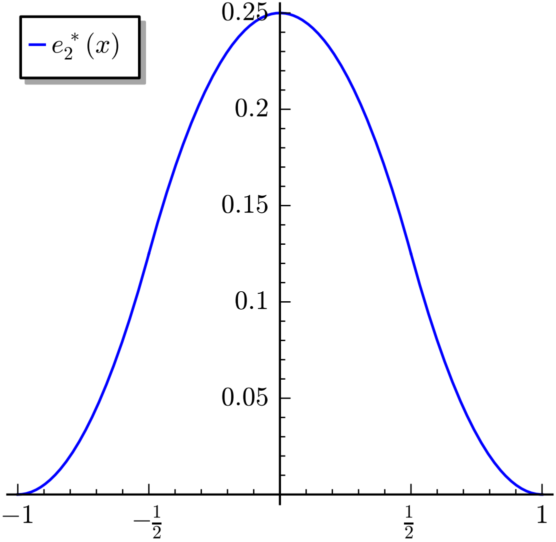
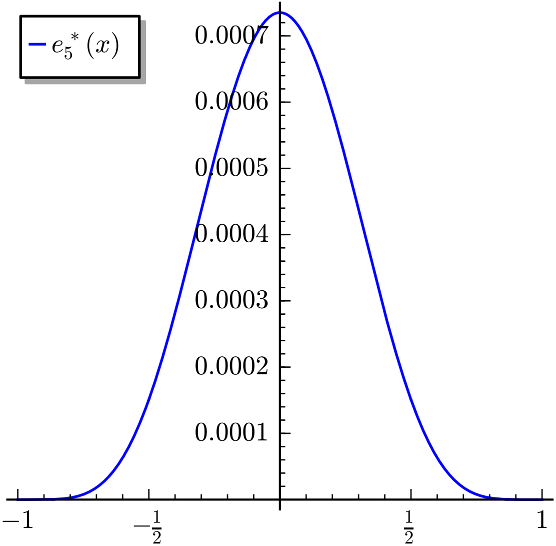


# Computing $e_r^*(x)$



# Boyanov's Spline

Boyanov discovered the "best" spline on $W^r$; that is, the spline $S(f;x)$ (capital 'S') such that,

$$e_r(S;x) = e_r^*(x),\ x \in [-1,1].$$

So, $S(f;x)$ has the best worst-case error at every point on our interval. However, the resulting formula is not so nice.

But it can still be expressed in our general form,

$$S(f;x) = \sum_{k=0}^{n} A_k(x) \cdot f^{(k)}(a) + B_k(x) \cdot f^{(k)}(b).$$

# Boyanov's Spline

We can apply Boyanov's spline to $e_r^*(x)$, noting that,

$$\left\{\frac{d^k}{dx^k}e_r^*\right\}(-1) = \left\{\frac{d^k}{dx^k}e_r^*\right\}(1) = 0, \;\; k = 0, 1, \ldots, r-1.$$

So,

$$S(e_r^*; x) = \sum_{k=0}^{n} A_k(x) \cdot 0 + B_k(x) \cdot 0 = 0.$$

Boyanov's spline applied to $e_r^*(x)$ is the zero function.

# Boyanov's Spline

Since $S(e_r^*; x) = 0$, the error of Boyanov's spline applied to $e_r^*(x)$ is,

$$|e_r^*(x) - S(e_r^*; x)| = |e_r^*(x) - 0| = e_r^*(x)$$

Recall that this is the maximal error that Boyanov's spline can achieve. Since $e_r^*$ itself induces this error, it is the worst function for Boyanov's spline.

# Optimal Error Bound

Using the formula we derived for $e_r^*(x)$, we can compute the optimal error bound which occurs at the midpoint.

We make a convenient definition,

$$g(r) = \left[1 + (-1)^{r+1}\right]/2 = \begin{cases} 0, & r \text{ even}, \\ 1, & r \text{ odd} \end{cases},$$

so that,

$$0 \in \left[\xi_{\frac{r-g(r)}{2}}, \xi_{\frac{r-g(r)+2}{2}}\right], \; r \geq 0.$$

# Optimal Error Bound

Now we can simply substitute this interval into the general piecewise formula. First, $e_1^*(0) = -1$. Then for $r > 1$,

$$r! \cdot e_r^*(0) = \sum_{k=0}^{\frac{r-g(r)}{2}-1} (-1)^{k+1} \left[ (\xi_{k+1})^r - (\xi_k)^r \right] + (-1)^{\frac{r-g(r)}{2}} \left( \xi_{\frac{r-g(r)}{2}} \right)^r$$

If the maximal error $e_r(s;x)$ of a spline $s(f;x)$ is less than (the norm of) this value, $s$ is optimal on $W^r$.

*Table:* $|e_r^*(0)|$ for certain values of $r$.

| $r$ | $\lvert e_r^*(0)\rvert$ |
|---|---|
| 1 | $1$ |
| 2 | $\frac{1}{4}$ |
| 3 | $\frac{2-\sqrt{2}}{12}$ |
| 4 | $\frac{3\sqrt{5}+8}{192}$ |
| 5 | $\frac{17-9\sqrt{3}}{1920}$ |

# Deriving a Bound on the Approximation Error

The exact Taylor expansion of a function $f$ about $a$ is,

$$\sum_{k=0}^{\mu-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{1}{(\mu-1)!}\int_a^x f^{(\mu)}(t)(x-t)^{\mu-1}\,dt$$

This can be understood as,

$$\text{(some polynomial)} + \text{(a remainder)}$$

and is *exactly* equal to the function $f$.

The barycentric coordinates of $x$ with respect to $-1$ and $1$ respectively are,

$$b_0\left(x\right) = \frac{1 - x}{2}$$

$$b_1\left(x\right) = \frac{x + 1}{2}$$

It follows from this definition that $b_0\left(x\right) + b_1\left(x\right) = 1$ for all $x$.

Now, assume that we have a spline method which reproduces polynomials of degree $\mu$.

We start by taking the exact Taylor expansion of $f$ about both endpoints. Call them $f_a(x)$ and $f_b(x)$. These two expansions are equal!

Next, we multiply $f_a(x)$ by $b_0(x)$ and $f_b(x)$ by $b_1(x)$. We do this to raise their degree by one. We want them to have degree $\mu$.

Since $f_a(x) = f_b(x)$,

$$
\begin{aligned}
& b_0(x) \cdot f_a(x) + b_1(x) \cdot f_b(x) \\
& = [b_0(x) + b_1(x)]\, f_a(x) \\
& = f(x)
\end{aligned}
$$

Since they're equal, we can instead apply our spline to the Taylor expansions. They are now of the form,

$$\text{some polynomial of degree } \mu + \text{remainder}$$

Since our spline reproduces polynomials of degree $\mu$, it will reproduce the polynomial part exactly. We are left with,

$$\begin{aligned} f(x) - s(f;x) &= \text{remainder1} + \text{remainder2} \\ &\quad - s(\text{remainder1}, x) - s(\text{remainder2}, x) \end{aligned}$$

And it turns out, this can be expressed as,

$$\int_{-1}^{1} f^{(\mu)}(t)\, e\left[E_{\mu}(t,x);x\right]\, dt,$$

where,

$$e(f;x) = f(x) - s(f;x)$$

and,

$$E_p(t,x) = \begin{cases} 0, & t \notin [-1,1], \\ b_0(x)\frac{(x-t)^{p-1}}{(p-1)!}, & t \in [-1,x], \\ -b_1(x)\frac{(x-t)^{p-1}}{(p-1)!}, & t \in [x,1]. \end{cases}$$

**Definition.** Let,

$$Q_p\left(t, x\right) = e\left[E_p\left(t, x\right); x\right]$$

Now we,

- Notice that the polynomial terms in $E_p$ are reproduced for $p \leq \mu$.
- Replace those polynomial terms with their approximations.
- Do lots of algebra.

To find...

# Deriving a Bound on the Approximation Error

$$Q_p(t,x) = \begin{cases} \sum\limits_{k=0}^{r-1} A_k \frac{(-1-t)^{p-k-1}}{(p-k-1)!}, & t \in [-1,x] \\ -\sum\limits_{k=0}^{r-1} B_k \frac{(1-t)^{p-k-1}}{(p-k-1)!}, & t \in [x,1] \end{cases}$$

Recall:

$$f(x) - s(f;x) = \int_{-1}^{1} f^{(\mu)}(t)\, Q_\mu(t,x)\, dt$$

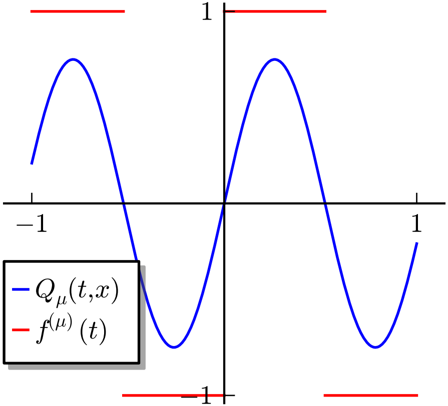
That means,

$$\begin{aligned} e_r(s;x) &= \sup_{f \in W^r} \left| f(x) - s(f;x) \right| \\ &= \sup_{f \in W^r} \left| \int_{-1}^{1} f^{(\mu)}(t)\, Q_\mu(t,x)\, dt \right| \end{aligned}$$

**Claim.**

$$\sup_{f \in W^r} \left| \int_{-1}^{1} f^{(\mu)}(t)\, Q_\mu(t,x)\, dt \right| = \int_{-1}^{1} \left| Q_p(t,x) \right| dt$$

**Proof.**



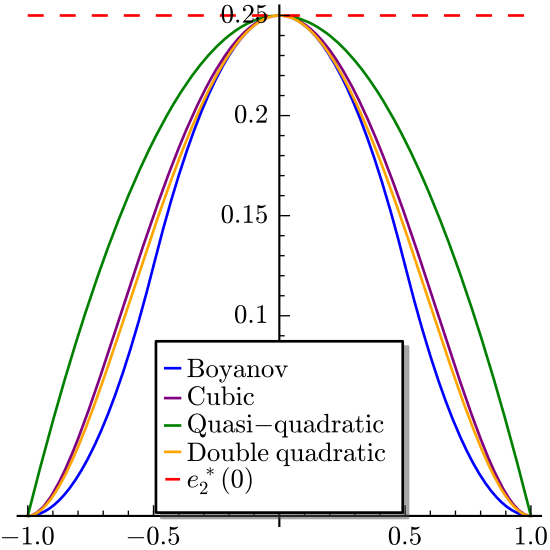

□

If we expand $Q_\mu(t,x)$ again, this gives us an expression for $e_r(s;x)$. This result was already known to Drs. Sorokina and Borodachov for $\mu = r-1$. After two changes of variable,

$$e_r(s;x) = \int_{-1-x}^{0} \left| \sum_{k=0}^{r-1} A_{r-k-1}(x) \frac{z^{(\mu-r+k)}}{(\mu-r+k)!} \right| dz$$

$$+ \int_{0}^{1-x} \left| \sum_{k=0}^{r-1} B_{r-k-1}(x) \frac{z^{(\mu-r+k)}}{(\mu-r+k)!} \right| dz$$

# Computed Error Bounds



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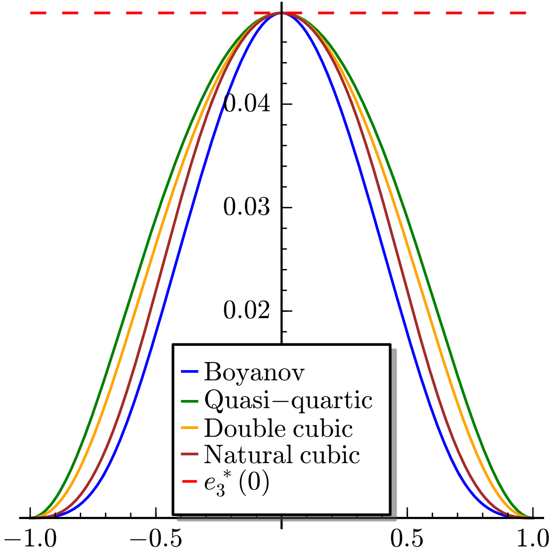


# Leibniz's Rule

We can also apply Leibniz's rule,

$$\frac{d}{dx}\int_{a(x)}^{b(x)} g\left(t,x\right)\,dt = \frac{d}{dx}\left\{b\left(x\right)\right\}\cdot g\left(b\left(x\right),x\right) - \frac{d}{dx}\left\{a\left(x\right)\right\}\cdot g\left(a\left(x\right),x\right) + \int_{a(x)}^{b(x)} \frac{d}{dx} g\left(t,x\right)\,dt$$

to $f(x) - s(f;x)$ directly.

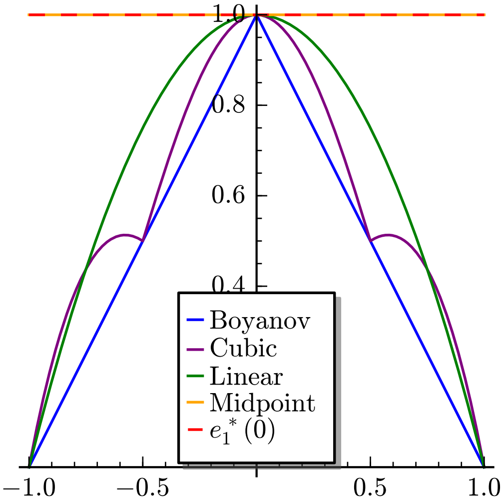
Since the left and right half of $Q(t, x)$ are equal at $t = x$, the,

$$\frac{d}{dx}\left\{f^{(u)}\left(x\right)\left[Q'_{\mu}\left(x,x\right) - Q'_{\mu}\left(x,x\right)\right]\right\}$$

term will cancel leaving us with,

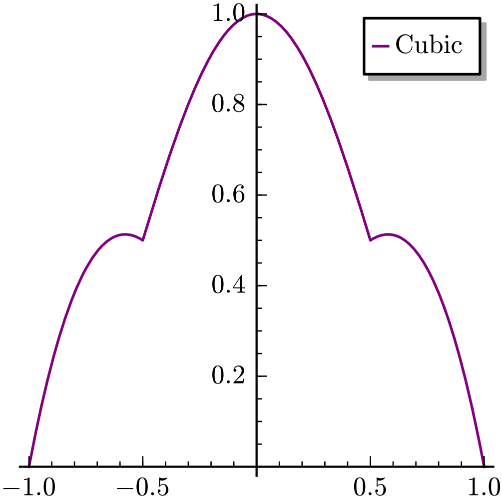
$$e'\left(f;x\right) = \int\limits_{-1}^{1} f^{(\mu)}\left(t\right)\frac{d}{dx}\left\{Q_{\mu}\left(t,x\right)\right\}$$

Unfortunately, we can't rely on this generally:

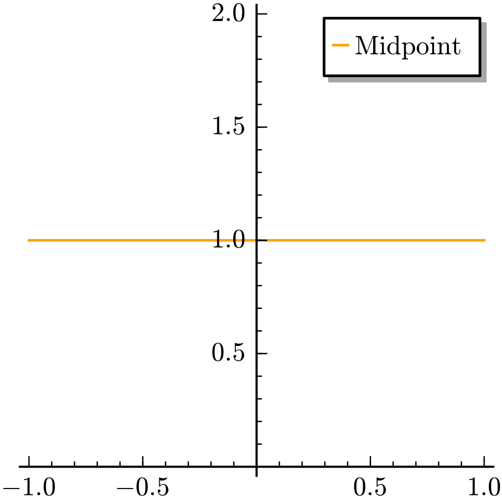


# Leibniz's Rule

The maximal error is not necessarily increasing on $[-1, 0]$.



And $e_r^*(0)$ can be achieved at points other than the midpoint.



# References

[1] Boyanov, B. D. Best Methods of Interpolation for Certain Classes of Differentiable Functions. Mathematical Notes, volume 17, issue 4, pp. 301-309. MAIK Nauka/Interperiodica, 1975.

[2] Powell, M. J. D. Approximation Theory and Methods. Cambridge University Press, Cambridge, 1981.