

*The uniqueness of Lyapunov rank among
symmetric cones*

Michael Orlitzky

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Background: Lyapunov rank

Problem (The LCP over \mathbb{R}_+^n).

$$\begin{aligned} & \text{find} && x \geq 0 \\ & \text{such that} && q + L(x) \geq 0 \\ & && \text{and } \langle q + L(x), x \rangle = 0 \end{aligned}$$

The condition $\langle q + L(x), x \rangle = 0$ is obviously necessary, so let's try to solve it independently.

A priori we have one equation

$$\langle q + L(x), x \rangle = 0$$

in the n variables x_1, x_2, \dots, x_n . But using $x \geq 0$ and $q + L(x) \geq 0$, it expands into n equations,

$$x_1 (q + L(x))_1 = 0$$

$$x_2 (q + L(x))_2 = 0$$

$$\vdots$$

$$x_n (q + L(x))_n = 0$$

Rationale.

More equations are usually better?

With n or more equations in n variables, we can apply Newton's method.

Problem (The LCP over a proper cone K).

With $y(x) := q + L(x)$,

$$\begin{aligned} & \text{find} && x \in K \\ & \text{such that} && y(x) \in K^* \\ & && \text{and } \langle y(x), x \rangle = 0 \end{aligned}$$

Question.

Can we still split the condition $\langle y(x), x \rangle = 0$ into a system equations?

Answer (RNPA¹ 2011).

Yes! ...but the system might have only one equation

The number of equations we get turns out to be a property of the cone.

¹ **Bilinear optimality constraints for the cone of positive polynomials.**
Rudolf, Noyan, Papp, and Alizadeh (2011).

In \mathbb{R}^n , $\langle x, y \rangle = 0$ splits into the system

$$\langle E_{11}x, y \rangle = 0$$

$$\langle E_{22}x, y \rangle = 0$$

$$\vdots$$

$$\langle E_{nn}x, y \rangle = 0$$

because

$$\langle E_{ii}x, y \rangle = 0 \text{ whenever } x, y \in \mathbb{R}_+^n \text{ and } \langle x, y \rangle = 0$$

In general? By analogy.

Definition.

A linear operator $L : V \rightarrow V$ is *Lyapunov-like* on K if $\langle L(x), y \rangle = 0$ whenever $(x, y) \in K \times K^*$ and $\langle x, y \rangle = 0$.

$\mathbf{LL}(K)$ is the set of all Lyapunov-like operators on K .

Now:

- $\mathbf{LL}(K)$ is a vector space (call its dimension β)
- There is a basis $\{L_1, L_2, \dots, L_\beta\}$ of $\mathbf{LL}(K)$
- $\langle x, y \rangle = 0$ splits into β equations,

$$\langle L_1(x), y \rangle = 0$$

$$\vdots$$

$$\langle L_\beta(x), y \rangle = 0$$

- $I \in \mathbf{LL}(K)$ corresponds to $\langle Ix, y \rangle = \langle x, y \rangle = 0$

Definition.

The number of equations we get,

$$\beta(K) := \dim(\mathbf{LL}(K)),$$

is called the *Lyapunov rank* of K .

Theorem (RNPA 2011).

If $\dim(K) = \dim(V) = n$, then the search space

$$\{(x, y) \in K \times K^* \mid \langle x, y \rangle = 0\}$$

is homeomorphic to \mathbb{R}^n .

In essence, we obtain $\beta(K)$ equations in n variables.

Moral. With $\beta(K)$, bigger is probably better.

Theorem (GT²O³ 2014–2017).

If K is a closed convex cone, then

$$\mathbf{LL}(K) = \text{Lie}(\text{Aut}(K))$$

and thus

$$\beta(K) = \dim(\text{Lie}(\text{Aut}(K))) = \text{mdim}(\text{Aut}(K))$$

² **On the bilinearity rank of a proper cone and Lyapunov-like transformations.** Gowda and Tao (2014).

³ **The Lyapunov rank of an improper cone.** Orlitzky (2017).

Background: Loewnerian cones

Definition.

A cone is *Loewnerian* if it is isomorphic to the real symmetric PSD cone, $\mathcal{H}_+^n(\mathbb{R})$.

They have applications to complementarity problems, if we can identify them⁴.

⁴ **Complementarity problems with respect to Loewnerian cones.** Seeger and Sossa (2014).

Question.

Can we use Lyapunov rank to determine whether or not a cone is Loewnerian?

Answer.

No, and this isn't too hard to show.

More generally, can *any* symmetric cones be uniquely identified using Lyapunov rank?

Background: symmetric cones

Definition.

A *symmetric cone* is a finite-dimensional, self-dual, homogeneous cone.

All symmetric cones K are isomorphic a unique direct sum of irreducible factors,

$$K \cong K_1 \oplus K_2 \oplus \cdots \oplus K_j$$

And (always up to isomorphism) there are only five families of those:

1. Lorentz cones \mathcal{L}_+^n for $n \geq 1$
2. Real PSD cones $\mathcal{H}_+^n(\mathbb{R})$ for $n \geq 3$
3. Complex PSD cones $\mathcal{H}_+^n(\mathbb{C})$ for $n \geq 3$
4. Quaternion PSD cones $\mathcal{H}_+^n(\mathbb{H})$ for $n \geq 3$
5. Whatever you call $\mathcal{H}_+^3(\mathbb{O}) := \{x^2 \mid x \in \mathcal{H}^3(\mathbb{O})\}$

Theorem (RNPA 2011).

If K and J are proper and if L^{-1} exists, then

$$\beta(L(K)) = \beta(K)$$

and

$$\beta(K \times J) = \beta(K) + \beta(J)$$

Thus:

$$\beta(K \oplus J) = \beta(K) + \beta(J)$$

Moral.

Knowing the Lyapunov ranks of those five families tells us the Lyapunov rank of any symmetric cone.

(At least in theory.)

Similacra: motivation

Question.

Suppose we are given a symmetric cone K . Does there exist another symmetric cone J such that

$$\dim (J) = \dim (K)$$

and

$$\beta (J) = \beta (K)$$

but

$$J \not\cong K?$$

Definition (signature, simulacrum).

The *signature* of K is $\sigma(K) := (\dim(K), \beta(K))$.

J is a *simulacrum* of K if $\sigma(K) = \sigma(J)$ and if K and J are non-isomorphic.

As shorthand we write $K \sim J$ (or $K \not\sim J$) to indicate that K and J are (not) simulacra.

Similacra: irreducible cones

Theorem (GT 2014, $n \geq 3$).

K	$\dim(K)$	$\beta(K)$
\mathcal{L}_+^n	n	$\frac{n^2 - n + 2}{2}$
$\mathcal{H}_+^n(\mathbb{R})$	$\frac{n^2 + n}{2}$	n^2
$\mathcal{H}_+^n(\mathbb{C})$	n^2	$2n^2 - 1$
$\mathcal{H}_+^n(\mathbb{H})$	$2n^2 - n$	$4n^2$
$\mathcal{H}_+^3(\mathbb{O})$	27	79

For convenience, we let

$$\mathbb{R}_+^n := \underbrace{\mathcal{L}_+^1 \oplus \mathcal{L}_+^1 \oplus \cdots \oplus \mathcal{L}_+^1}_{n \text{ times}}$$

and append an entry to the table ($n \geq 0$):

K	$\dim(K)$	$\beta(K)$
\mathbb{R}_+^n	n	n

Proposition.

Among non-isomorphic n -dimensional symmetric cones, the Lyapunov rank of \mathcal{L}_+^n is strictly maximal.

Proof.

If $n \leq 2$, everyone is isomorphic. If $n \geq 3$ then

$$\beta(\mathcal{L}_+^n) - \beta(\mathcal{L}_+^k \oplus \mathcal{L}_+^{n-k}) = (n-k)k - 1 > 0$$

so using two or more Lorentz cones won't work.

Proof (cont'd).

For the others:

$$\beta\left(\mathcal{L}_+^{(m^2+m)/2}\right) - \beta\left(\mathcal{H}_+^m(\mathbb{R})\right) = \frac{1}{8}m^4 + \frac{1}{4}m^3 - \mathcal{O}(m^2)$$

$$\beta\left(\mathcal{L}_+^{m^2}\right) - \beta\left(\mathcal{H}_+^m(\mathbb{C})\right) = \frac{1}{2}m^4 - \mathcal{O}(m^2)$$

$$\beta\left(\mathcal{L}_+^{2m^2-m}\right) - \beta\left(\mathcal{H}_+^m(\mathbb{H})\right) = 2m^4 - \mathcal{O}(m^3)$$

$$\beta\left(\mathcal{L}_+^{27}\right) - \beta\left(\mathcal{H}_+^3(\mathbb{O})\right) = 273$$

Proof (cont'd).

At $m = 3$, the higher-order (positive) terms already dominate the lower-order (negative) ones.

Now:

- All symmetric cones are made up of factors
- Other factors do worse than Lorentz cones
- Two or more Lorentz cones are worse than one



Corollary.

\mathcal{L}_+^n has no symmetric similacra for any $n \geq 0$.

Proof.

No other symmetric cone of the same dimension has a large enough Lyapunov rank. □

Proposition.

$\mathcal{H}_+^n(\mathbb{R})$ has symmetric similacra for all $n \geq 3$.

Proof.

$$\mathcal{H}_+^n(\mathbb{R}) \sim \mathcal{L}_+^{n+1} \oplus \mathbb{R}_+^{(n^2-n-2)/2}$$



Corollary.

Lyapunov rank cannot be used to identify
Loewnerian cones.

Proposition.

$\mathcal{H}_+^3(\mathbb{C})$ has no symmetric similacra.

Proof.

Only $\mathcal{H}_+^n(\mathbb{R})$ and \mathcal{L}_+^n are small enough to work.

By the previous result, we need only try \mathcal{L}_+^n factors.
And $\beta(\mathcal{L}_+^n)$ is too large for $n \geq 6$, so we may assume
that $n \leq 5$.

Proof (cont'd).

If \mathcal{L}_+^5 is a factor, the rest have signature $(4, 6)$. The Lyapunov ranks of \mathbb{R}_+^4 and $\mathcal{L}_+^3 \oplus \mathbb{R}_+^1$ are too small, and that of \mathcal{L}_+^4 is too large. Thus \mathcal{L}_+^5 is not a factor.

If \mathcal{L}_+^4 is a factor, then the rest have signature $(5, 10)$. \mathcal{L}_+^5 has already been ruled out. The Lyapunov rank of $\mathcal{L}_+^4 \oplus \mathbb{R}_+^1$ is too small, and further splitting of \mathcal{L}_+^4 only makes things worse, so \mathcal{L}_+^4 is not a factor.

Finally, $\beta(\mathcal{L}_+^3 \oplus \mathcal{L}_+^3 \oplus \mathcal{L}_+^3)$ is too small. □

Proposition.

$\mathcal{H}_+^n(\mathbb{C})$ has symmetric similacra for all $n \geq 4$.

Proof.

$$\mathcal{H}_+^4(\mathbb{C}) \sim \mathcal{L}_+^5 \oplus \mathcal{L}_+^5 \oplus \mathcal{L}_+^4 \oplus \mathbb{R}_+^2$$

and if $n \geq 5$, then

$$\mathcal{H}_+^n(\mathbb{C}) \sim \mathcal{L}_+^{n+1} \oplus \mathcal{L}_+^{n+1} \oplus \mathbb{R}_+^{(n^2-5n+1)} \oplus \underbrace{\mathcal{L}_+^3 \oplus \dots \oplus \mathcal{L}_+^3}_{n-1 \text{ times}}$$



Proposition.

$\mathcal{H}_+^n(\mathbb{H})$ has symmetric similacra for all $n \geq 3$.

Proof.

$$\mathcal{H}_+^3(\mathbb{H}) \sim \mathcal{L}_+^8 \oplus \mathbb{R}_+^7$$

$$\mathcal{H}_+^4(\mathbb{H}) \sim \mathcal{L}_+^{10} \oplus \mathbb{R}_+^{18}$$

$$\mathcal{H}_+^5(\mathbb{H}) \sim \mathcal{L}_+^{12} \oplus \mathbb{R}_+^{33}$$

and if $n \geq 6$, then

$$\mathcal{H}_+^n(\mathbb{H}) \sim \mathcal{H}_+^{n+1}(\mathbb{C}) \oplus \mathcal{L}_+^{n+1} \oplus \mathcal{L}_+^{n+1} \oplus \mathbb{R}_+^{(n^2-5n-3)}$$



Proposition.

$\mathcal{H}_+^3(\mathbb{O})$ has symmetric similacra.

Proof.

$$\mathcal{H}_+^3(\mathbb{O}) \sim \mathcal{L}_+^{11} \oplus \mathcal{L}_+^5 \oplus \mathcal{L}_+^3 \oplus \mathbb{R}_+^8$$



Theorem.

$\mathcal{H}_+^3(\mathbb{C})$ and \mathcal{L}_+^n are the only irreducible symmetric cones without symmetric similacra.

Similacra: reducible cones

Lemma.

If a symmetric cone contains more than one $\mathcal{H}_+^3(\mathbb{C})$ factor, then it has symmetric similacra.

Proof.

$$\mathcal{H}_+^3(\mathbb{C}) \oplus \mathcal{H}_+^3(\mathbb{C}) \sim \mathcal{L}_+^7 \oplus \mathcal{L}_+^3 \oplus \mathbb{R}_+^8$$

$$\mathcal{H}_+^3(\mathbb{C}) \oplus \mathcal{H}_+^3(\mathbb{C}) \oplus \mathcal{H}_+^3(\mathbb{C}) \sim \mathcal{L}_+^8 \oplus \mathcal{L}_+^4 \oplus \mathbb{R}_+^{15}$$

If there are $m > 1$ factors, $m = 2k + 3j$ for some k, j . □

Theorem.

If K is a symmetric cone and if K has no symmetric similacra, then either

$$K \cong \mathcal{L}_+^{n_1} \oplus \mathcal{L}_+^{n_2} \oplus \cdots \oplus \mathcal{L}_+^{n_k}$$

or

$$K \cong \mathcal{H}_+^3(\mathbb{C}) \oplus \mathcal{L}_+^{n_1} \oplus \mathcal{L}_+^{n_2} \oplus \cdots \oplus \mathcal{L}_+^{n_k}$$

for $k, n_i \in \mathbb{N}$.

Question.

Starting with $K = \mathcal{H}_+^3(\mathbb{C})$, we have no symmetric similacra. But as we just saw, we cannot add more $\mathcal{H}_+^3(\mathbb{C})$ factors without acquiring some.

Is the same true of \mathcal{L}_+^n factors? Does $\mathcal{H}_+^3(\mathbb{C}) \oplus \mathcal{L}_+^n$ have symmetric similacra? For which n ?

Lemma.

If $n \geq 10$, then $\mathcal{H}_+^3(\mathbb{C}) \oplus \mathcal{L}_+^n \simeq J \oplus \mathcal{L}_+^{n+k}$ for any symmetric J and $k \geq 0$.

Proof. ($k = 0$ is trivial)

The smallest possible Lyapunov rank we can make with $k \geq 1$ is at $k = 1$ with $J = \mathbb{R}_+^8$. But if $n \geq 10$ then

$$\beta(\mathbb{R}_+^8 \oplus \mathcal{L}_+^{n+1}) - \beta(\mathcal{H}_+^3(\mathbb{C}) \oplus \mathcal{L}_+^n) = n - 9 \geq 1$$



Lemma.

If $n \geq 31$, then $\mathcal{H}_+^3(\mathbb{C}) \oplus \mathcal{L}_+^n \not\cong \mathcal{L}_+^{9+k} \oplus \mathcal{L}_+^{n-k}$ for any $k \geq 1$.

Proof.

If $k \geq n - 9$, then \mathcal{L}_+^{9+k} cannot be a factor by the previous lemma. Thus we may assume that $k < n - 9$.

Proof (cont'd).

Given that $k < n - 9$, the difference

$$\begin{aligned} \beta(\mathcal{H}_+^3(\mathbb{C}) \oplus \mathcal{L}_+^n) - \beta(\mathcal{L}_+^{9+k} \oplus \mathcal{L}_+^{n-k}) \\ = \\ k[(n-9) - k] - 20 \end{aligned}$$

as a function of k is a downwards parabola. Checking the endpoints with $n \geq 31$, it is minimized at $k = 1$ with value $n - 30 > 0$.



Conjecture (checked up to $n = 200$).

$\mathcal{H}_+^3(\mathbb{C}) \oplus \mathcal{L}_+^n$ has no symmetric similacra when $n \geq 31$.

Rationale.

If the preceding lemma extends to more than two factors, then both \mathcal{L}_+^{n+k} and \mathcal{L}_+^{n-k} will be prohibited in a similacrum when $n \geq 31$. By an earlier result, it suffices to rule out Lorentz cone factors.

Up to $n = 30$, we can just check:

n	K	\sim
2	$\mathcal{H}_+^3(\mathbb{C}) \oplus \mathcal{L}_+^2$	$\mathcal{L}_+^5 \oplus \mathcal{L}_+^3 \oplus \mathcal{L}_+^3$
3	$\mathcal{H}_+^3(\mathbb{C}) \oplus \mathcal{L}_+^3$	$\mathcal{L}_+^4 \oplus \mathcal{L}_+^4 \oplus \mathcal{L}_+^4$
4	$\mathcal{H}_+^3(\mathbb{C}) \oplus \mathcal{L}_+^4$	$\mathcal{L}_+^6 \oplus \mathcal{L}_+^3 \oplus \mathbb{R}_+^4$
5	$\mathcal{H}_+^3(\mathbb{C}) \oplus \mathcal{L}_+^5$	$\mathcal{L}_+^6 \oplus \mathcal{L}_+^4 \oplus \mathcal{L}_+^3 \oplus \mathbb{R}_+^1$

n	K	\sim
6	$\mathcal{H}_+^3(\mathbb{C}) \oplus \mathcal{L}_+^6$	$\mathcal{L}_+^5 \oplus \mathcal{L}_+^5 \oplus \mathcal{L}_+^5$
7	$\mathcal{H}_+^3(\mathbb{C}) \oplus \mathcal{L}_+^7$	$\mathcal{L}_+^6 \oplus \mathcal{L}_+^6 \oplus \mathcal{L}_+^4$
8	$\mathcal{H}_+^3(\mathbb{C}) \oplus \mathcal{L}_+^8$	$\mathcal{L}_+^9 \oplus \mathcal{L}_+^3 \oplus \mathbb{R}_+^5$
9	$\mathcal{H}_+^3(\mathbb{C}) \oplus \mathcal{L}_+^9$	$\mathcal{L}_+^{10} \oplus \mathbb{R}_+^8$
10	$\mathcal{H}_+^3(\mathbb{C}) \oplus \mathcal{L}_+^{10}$	$\mathcal{L}_+^9 \oplus \mathcal{L}_+^7 \oplus \mathcal{L}_+^3$

n	K	\sim
15	$\mathcal{H}_+^3(\mathbb{C}) \oplus \mathcal{L}_+^{15}$	$\mathcal{L}_+^{14} \oplus \mathcal{L}_+^8 \oplus \mathbb{R}_+^2$
18	$\mathcal{H}_+^3(\mathbb{C}) \oplus \mathcal{L}_+^{18}$	$\mathcal{L}_+^{14} \oplus \mathcal{L}_+^{13}$
21	$\mathcal{H}_+^3(\mathbb{C}) \oplus \mathcal{L}_+^{21}$	$\mathcal{L}_+^{19} \oplus \mathcal{L}_+^{11}$
22	$\mathcal{H}_+^3(\mathbb{C}) \oplus \mathcal{L}_+^{22}$	$\mathcal{L}_+^{21} \oplus \mathcal{L}_+^9 \oplus \mathbb{R}_+^1$
30	$\mathcal{H}_+^3(\mathbb{C}) \oplus \mathcal{L}_+^{30}$	$\mathcal{L}_+^{29} \oplus \mathcal{L}_+^{10}$

Conjecture.

$\mathcal{H}_+^3(\mathbb{C}) \oplus \mathcal{L}_+^n$ has symmetric similacra if and only if $n \in \{2, 3, \dots, 10, 15, 18, 21, 22, 30\}$.

Conjecture.

If K is a symmetric cone with an $\mathcal{H}_+^3(\mathbb{C})$ factor and no symmetric similacra, then

$$K \cong \mathcal{H}_+^3(\mathbb{C}) \oplus \mathcal{L}_+^{n_1} \oplus \mathcal{L}_+^{n_2} \oplus \dots \oplus \mathcal{L}_+^{n_k}$$

where $k \in \mathbb{N}$ and $n_i \notin \{2, 3, \dots, 10, 15, 18, 21, 22, 30\}$.

The end