The uniqueness of Lyapunov rank among symmetric cones

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Workshop on Variational Analysis and Euclidean Jordan Algebras
Rancagua, September 26–27, 2024

Background: Lyapunov rank

Problem (The LCP over $\mathbb{R}^n_+$).

$$\begin{aligned} \text{find} \quad & x \geq 0 \\ \text{such that} \quad & q + L(x) \geq 0 \\ \text{and} \quad & \langle q + L(x), x \rangle = 0 \end{aligned}$$

The condition $\langle q + L(x), x \rangle = 0$ is obviously necessary, so let's try to solve it independently.

A priori we have one equation

$$\langle q + L(x), x \rangle = 0$$

in the n variables $x_1, x_2, \ldots, x_n$. But using $x \geq 0$ and $q + L(x) \geq 0$, it expands into n equations,

$$\begin{aligned} x_1 \left(q + L(x)\right)_1 &= 0 \\ x_2 \left(q + L(x)\right)_2 &= 0 \\ &\vdots \\ x_n \left(q + L(x)\right)_n &= 0 \end{aligned}$$

Rationale.

More equations are usually better?

With n or more equations in n variables, we can apply Newton's method.

Problem (The LCP over a proper cone K).

With $y(x) \coloneqq q + L(x)$,

$$\begin{aligned} \text{find} \quad & x \in K \\ \text{such that} \quad & y(x) \in K^{*} \\ \text{and } & \langle y(x), x \rangle = 0 \end{aligned}$$

Question.

Can we still split the condition $\langle y(x), x \rangle = 0$ into a system equations?

Answer (RNPA[1] 2011).

Yes! …but the system might have only one equation

The number of equations we get turns out to be a property of the cone.

[1] **Bilinear optimality constraints for the cone of positive polynomials**. Rudolf, Noyan, Papp, and Alizadeh (2011).

In $\mathbb{R}^n$, $\langle x, y \rangle = 0$ splits into the system

$$\begin{aligned} \langle E_{11} x, y \rangle &= 0 \\ \langle E_{22} x, y \rangle &= 0 \\ &\vdots \\ \langle E_{nn} x, y \rangle &= 0 \end{aligned}$$

because

$$\langle E_{ii} x, y \rangle = 0 \text{ whenever } x, y \in \mathbb{R}^n_+ \text{ and } \langle x, y \rangle = 0$$

In general? By analogy.

Definition.

A linear operator $L : V \to V$ is *Lyapunov-like* on K if $\langle L(x), y \rangle = 0$ whenever $(x, y) \in K \times K^*$ and $\langle x, y \rangle = 0$.

$\mathbf{LL}(K)$ is the set of all Lyapunov-like operators on K.

Now:

- $\mathbf{LL}(K)$ is a vector space (call its dimension β)
- There is a basis $\{L_1, L_2, \ldots, L_\beta\}$ of $\mathbf{LL}(K)$
- $\langle x, y \rangle = 0$ splits into β equations,

$$\begin{gathered} \langle L_1(x), y \rangle = 0 \\ \vdots \\ \langle L_\beta(x), y \rangle = 0 \end{gathered}$$

- $I \in \mathbf{LL}(K)$ corresponds to $\langle Ix, y \rangle = \langle x, y \rangle = 0$

Definition.

The number of equations we get,

$$\beta(K) \coloneqq \dim(\mathbf{LL}(K)),$$

is called the *Lyapunov rank* of K.

Theorem (RNPA 2011).

If $\dim(K) = \dim(V) = n$, then the search space

$$\left\{(x, y) \in K \times K^{*} \mid \langle x, y \rangle = 0\right\}$$

is homeomorphic to $\mathbb{R}^{n}$.

In essence, we obtain $\beta(K)$ equations in n variables.

Moral. With $\beta(K)$, bigger is probably better.

Theorem (GT[2]O[3] 2014–2017).

If K is a closed convex cone, then

$$\mathbf{LL}(K) = \mathrm{Lie}(\mathrm{Aut}(K))$$

and thus

$$\beta(K) = \dim(\mathrm{Lie}(\mathrm{Aut}(K))) = \mathrm{mdim}(\mathrm{Aut}(K))$$

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[2] **On the bilinearity rank of a proper cone and Lyapunov-like transformations**. Gowda and Tao (2014).

[3] **The Lypaunov rank of an improper cone**. Orlitzky (2017).

Background: Loewnerian cones

Definition.

A cone is *Loewnerian* if it is isomorphic to the real symmetric PSD cone, $\mathcal{H}_{+}^{n}(\mathbb{R})$.

They have applications to complementarity problems, if we can identify them[4].

[4] **Complementarity problems with respect to Loewnerian cones**. Seeger and Sossa (2014).

Question.

Can we use Lypaunov rank to determine whether or not a cone is Loewnerian?

Answer.

No, and this isn't too hard to show.

More generally, can *any* symmetric cones be uniquely identified using Lypaunov rank?

Background: symmetric cones

Definition.

A *symmetric cone* is a finite-dimensional, self-dual, homogeneous cone.

All symmetric cones K are isomorphic a unique direct sum of irreducible factors,

$$K \cong K_1 \oplus K_2 \oplus \cdots \oplus K_j$$

And (always up to isomorphism) there are only five families of those:

1. Lorentz cones $\mathcal{L}_+^n$ for $n \geq 1$
2. Real PSD cones $\mathcal{H}_+^n(\mathbb{R})$ for $n \geq 3$
3. Complex PSD cones $\mathcal{H}_+^n(\mathbb{C})$ for $n \geq 3$
4. Quaternion PSD cones $\mathcal{H}_+^n(\mathbb{H})$ for $n \geq 3$
5. Whatever you call $\mathcal{H}_+^3(\mathbb{O}) \coloneqq \left\{x^2 \mid x \in \mathcal{H}^3(\mathbb{O})\right\}$

Theorem (RNPA 2011).

If K and J are proper and if L^{-1} exists, then

$$\beta(L(K)) = \beta(K)$$

and

$$\beta(K \times J) = \beta(K) + \beta(J)$$

Thus:

$$\beta(K \oplus J) = \beta(K) + \beta(J)$$

Moral.

Knowing the Lyapunov ranks of those five families tells us the Lyapunov rank of any symmetric cone.

(At least in theory.)

Similacra: motivation

Question.

Suppose we are given a symmetric cone K. Does there exist another symmetric cone J such that

$$\dim(J) = \dim(K)$$

and

$$\beta(J) = \beta(K)$$

but

$$J \ncong K?$$

Definition (signature, similacrum).

The *signature* of K is $\sigma(K) \coloneqq (\dim(K), \beta(K))$.

J is a *similacrum* of K if $\sigma(K) = \sigma(J)$ and if K and J are non-isomorphic.

As shorthand we write $K \sim J$ (or $K \nsim J$) to indicate that K and J are (not) similacra.

Similacra: irreducible cones

Theorem (GT 2014, $n \geq 2$).

K	$\dim(K)$	$\beta(K)$
$\mathcal{L}^n_+$	n	$\frac{n^2-n+2}{2}$
$\mathcal{H}^n_+(\mathbb{R})$	$\frac{n^2+n}{2}$	n^2
$\mathcal{H}^n_+(\mathbb{C})$	n^2	$2n^2-1$
$\mathcal{H}^n_+(\mathbb{H})$	$2n^2-n$	$4n^2$
$\mathcal{H}^3_+(\mathbb{O})$	27	79

For convenience, we let

$$\mathbb{R}^n_+ := \underbrace{\mathcal{L}^1_+ \oplus \mathcal{L}^1_+ \oplus \cdots \oplus \mathcal{L}^1_+}_{n \text{ times}}$$

and append an entry to the table ($n \geq 0$):

K	$\dim(K)$	$\beta(K)$
$\mathbb{R}^n_+$	n	n

Proposition.

Among non-isomorphic n-dimensional symmetric cones, the Lyapunov rank of $\mathcal{L}_{+}^{n}$ is strictly maximal.

Proof.

If $n \le 2$, everyone is isomorphic. If $n \ge 3$ then

$$\beta\left(\mathcal{L}_{+}^{n}\right) - \beta\left(\mathcal{L}_{+}^{k} \oplus \mathcal{L}_{+}^{n-k}\right) = (n-k)\,k - 1 > 0$$

so using two or more Lorentz cones won't work.

Proof (cont'd).

For the others:

$$\beta\left(\mathcal{L}_{+}^{(m^2+m)/2}\right)-\beta\left(\mathcal{H}_{+}^{m}\left(\mathbb{R}\right)\right)=\frac{1}{8}m^4+\frac{1}{4}m^3-\mathcal{O}\left(m^2\right)$$
$$\beta\left(\mathcal{L}_{+}^{m^2}\right)-\beta\left(\mathcal{H}_{+}^{m}\left(\mathbb{C}\right)\right)=\frac{1}{2}m^4-\mathcal{O}\left(m^2\right)$$
$$\beta\left(\mathcal{L}_{+}^{2m^2-m}\right)-\beta\left(\mathcal{H}_{+}^{m}\left(\mathbb{H}\right)\right)=2m^4-\mathcal{O}\left(m^3\right)$$
$$\beta\left(\mathcal{L}_{+}^{27}\right)-\beta\left(\mathcal{H}_{+}^{3}\left(\mathbb{O}\right)\right)=273$$

Proof (cont'd).

At $m = 3$, the higher-order (positive) terms already dominate the lower-order (negative) ones.

Now:

- All symmetric cones are made up of factors
- Other factors do worse than Lorentz cones
- Two or more Lorentz cones are worse than one

□

Corollary.

$\mathcal{L}_{+}^{n}$ has no symmetric similacra for any $n \geq 0$.

Proof.

No other symmetric cone of the same dimension has a large enough Lyapunov rank. $\square$

Proposition.

$\mathcal{H}^n_+(\mathbb{R})$ has symmetric similacra for all $n \geq 3$.

Proof.

$$\mathcal{H}^n_+(\mathbb{R}) \sim \mathcal{L}^{n+1}_+ \oplus \mathbb{R}^{(n^2-n-2)/2}_+$$

□

Corollary.

Lyapunov rank cannot be used to identify Loewnerian cones.

Proposition.

$\mathcal{H}^3_+(\mathbb{C})$ has no symmetric similacra.

Proof.

Only $\mathcal{H}^n_+(\mathbb{R})$ and $\mathcal{L}^n_+$ are small enough to work.

By the previous result, we need only try $\mathcal{L}^n_+$ factors. And $\beta(\mathcal{L}^n_+)$ is too large for $n \geq 6$, so we may assume that $n \leq 5$.

Proof (cont'd).

If $\mathcal{L}^5_+$ is a factor, the rest have signature $(4,6)$. The Lyapunov ranks of $\mathbb{R}^4_+$ and $\mathcal{L}^3_+ \oplus \mathbb{R}^1_+$ are too small, and that of $\mathcal{L}^4_+$ is too large. Thus $\mathcal{L}^5_+$ is not a factor.

If $\mathcal{L}^4_+$ is a factor, then the rest have signature $(5,10)$. $\mathcal{L}^5_+$ has already been ruled out. The Lyapunov rank of $\mathcal{L}^4_+ \oplus \mathbb{R}^1_+$ is too small, and further splitting of $\mathcal{L}^4_+$ only makes things worse, so $\mathcal{L}^4_+$ is not a factor.

Finally, $\beta\left(\mathcal{L}^3_+ \oplus \mathcal{L}^3_+ \oplus \mathcal{L}^3_+\right)$ is too small. □

Proposition.

$\mathcal{H}_{+}^{n}(\mathbb{C})$ has symmetric similacra for all $n \geq 4$.

Proof.

$$\mathcal{H}_{+}^{4}(\mathbb{C}) \sim \mathcal{L}_{+}^{5} \oplus \mathcal{L}_{+}^{5} \oplus \mathcal{L}_{+}^{4} \oplus \mathbb{R}_{+}^{2}$$

and if $n \geq 5$, then

$$\mathcal{H}_{+}^{n}(\mathbb{C}) \sim \mathcal{L}_{+}^{n+1} \oplus \mathcal{L}_{+}^{n+1} \oplus \mathbb{R}_{+}^{(n^2-5n+1)} \oplus \underbrace{\mathcal{L}_{+}^{3} \oplus \cdots \oplus \mathcal{L}_{+}^{3}}_{n-1 \text{ times}}$$

□

Proposition.

$\mathcal{H}^n_+(\mathbb{H})$ has symmetric similacra for all $n \geq 3$.

Proof.

$$\mathcal{H}^3_+(\mathbb{H}) \sim \mathcal{L}^8_+ \oplus \mathbb{R}^7_+$$
$$\mathcal{H}^4_+(\mathbb{H}) \sim \mathcal{L}^{10}_+ \oplus \mathbb{R}^{18}_+$$
$$\mathcal{H}^5_+(\mathbb{H}) \sim \mathcal{L}^{12}_+ \oplus \mathbb{R}^{33}_+$$

and if $n \geq 6$, then

$$\mathcal{H}^n_+(\mathbb{H}) \sim \mathcal{H}^{n+1}_+(\mathbb{C}) \oplus \mathcal{L}^{n+1}_+ \oplus \mathcal{L}^{n+1}_+ \oplus \mathbb{R}^{(n^2-5n-3)}_+$$

□

Proposition.

$\mathcal{H}^3_+(\mathbb{O})$ has symmetric similacra.

Proof.

$$\mathcal{H}^3_+(\mathbb{O}) \sim \mathcal{L}^{11}_+ \oplus \mathcal{L}^5_+ \oplus \mathcal{L}^3_+ \oplus \mathbb{R}^8_+$$

□

Theorem.

$\mathcal{H}_{+}^{3}(\mathbb{C})$ and $\mathcal{L}_{+}^{n}$ are the only irreducible symmetric cones without symmetric similacra.

Similacra: reducible cones

Lemma.

If a symmetric cone contains more than one $\mathcal{H}^3_+(\mathbb{C})$ factor, then it has symmetric similacra.

Proof.

$$\mathcal{H}^3_+(\mathbb{C}) \oplus \mathcal{H}^3_+(\mathbb{C}) \sim \mathcal{L}^7_+ \oplus \mathcal{L}^3_+ \oplus \mathbb{R}^8_+$$
$$\mathcal{H}^3_+(\mathbb{C}) \oplus \mathcal{H}^3_+(\mathbb{C}) \oplus \mathcal{H}^3_+(\mathbb{C}) \sim \mathcal{L}^8_+ \oplus \mathcal{L}^4_+ \oplus \mathbb{R}^{15}_+$$

If there are $m > 1$ factors, $m = 2k + 3j$ for some k, j. $\square$

Theorem.

If K is a symmetric cone and if K has no symmetric similacra, then either

$$K \cong \mathcal{L}_{+}^{n_1} \oplus \mathcal{L}_{+}^{n_2} \oplus \cdots \oplus \mathcal{L}_{+}^{n_k}$$

or

$$K \cong \mathcal{H}_{+}^{3}\left(\mathbb{C}\right) \oplus \mathcal{L}_{+}^{n_1} \oplus \mathcal{L}_{+}^{n_2} \oplus \cdots \oplus \mathcal{L}_{+}^{n_k}$$

for $k, n_i \in \mathbb{N}$.

Question.

Starting with $K = \mathcal{H}_{+}^{3}(\mathbb{C})$, we have no symmetric similacra. But as we just saw, we cannot add more $\mathcal{H}_{+}^{3}(\mathbb{C})$ factors without acquiring some.

Is the same true of $\mathcal{L}_{+}^{n}$ factors? Does $\mathcal{H}_{+}^{3}(\mathbb{C}) \oplus \mathcal{L}_{+}^{n}$ have symmetric similacra? For which n?

Lemma.

If $n \geq 10$, then $\mathcal{H}^3_+(\mathbb{C}) \oplus \mathcal{L}^n_+ \not\sim J \oplus \mathcal{L}^{n+k}_+$ for any symmetric J and $k \geq 0$.

Proof. ($k = 0$ is trivial)

The smallest possible Lyapunov rank we can make with $k \geq 1$ is at $k = 1$ with $J = \mathbb{R}^8_+$. But if $n \geq 10$ then

$$\beta\left(\mathbb{R}^8_+ \oplus \mathcal{L}^{n+1}_+\right) - \beta\left(\mathcal{H}^3_+(\mathbb{C}) \oplus \mathcal{L}^n_+\right) = n - 9 \geq 1$$

□

Lemma.

If $n \geq 31$, then $\mathcal{H}_{+}^{3}(\mathbb{C}) \oplus \mathcal{L}_{+}^{n} \not\sim \mathcal{L}_{+}^{9+k} \oplus \mathcal{L}_{+}^{n-k}$ for any $k \geq 1$.

Proof.

If $k \geq n - 9$, then $\mathcal{L}_{+}^{9+k}$ cannot be a factor by the previous lemma. Thus we may assume that $k < n - 9$.

Proof (cont'd).

Given that $k < n - 9$, the difference

$$\beta\left(\mathcal{H}_{+}^{3}(\mathbb{C}) \oplus \mathcal{L}_{+}^{n}\right) - \beta\left(\mathcal{L}_{+}^{9+k} \oplus \mathcal{L}_{+}^{n-k}\right)$$
$$=$$
$$k\left[(n-9) - k\right] - 20$$

as a function of k is a downwards parabola. Checking the endpoints with $n \geq 31$, it is minimized at $k = 1$ with value $n - 30 > 0$.

□

Conjecture (checked up to $n = 200$).

$\mathcal{H}^3_+(\mathbb{C}) \oplus \mathcal{L}^n_+$ has no symmetric similacra when $n \geq 31$.

Rationale.

If the preceding lemma extends to more than two factors, then both $\mathcal{L}^{n+k}_+$ and $\mathcal{L}^{n-k}_+$ will be prohibited in a similacrum when $n \geq 31$. By an earlier result, it suffices to rule out Lorentz cone factors.

Up to $n = 30$, we can just check:

n	K	$\sim$
2	$\mathcal{H}^3_+(\mathbb{C}) \oplus \mathcal{L}^2_+$	$\mathcal{L}^5_+ \oplus \mathcal{L}^3_+ \oplus \mathcal{L}^3_+$
3	$\mathcal{H}^3_+(\mathbb{C}) \oplus \mathcal{L}^3_+$	$\mathcal{L}^4_+ \oplus \mathcal{L}^4_+ \oplus \mathcal{L}^4_+$
4	$\mathcal{H}^3_+(\mathbb{C}) \oplus \mathcal{L}^4_+$	$\mathcal{L}^6_+ \oplus \mathcal{L}^3_+ \oplus \mathbb{R}^4_+$
5	$\mathcal{H}^3_+(\mathbb{C}) \oplus \mathcal{L}^5_+$	$\mathcal{L}^6_+ \oplus \mathcal{L}^4_+ \oplus \mathcal{L}^3_+ \oplus \mathbb{R}^1_+$

n	K	$\sim$
6	$\mathcal{H}^{3}_{+}(\mathbb{C}) \oplus \mathcal{L}^{6}_{+}$	$\mathcal{L}^{5}_{+} \oplus \mathcal{L}^{5}_{+} \oplus \mathcal{L}^{5}_{+}$
7	$\mathcal{H}^{3}_{+}(\mathbb{C}) \oplus \mathcal{L}^{7}_{+}$	$\mathcal{L}^{6}_{+} \oplus \mathcal{L}^{6}_{+} \oplus \mathcal{L}^{4}_{+}$
8	$\mathcal{H}^{3}_{+}(\mathbb{C}) \oplus \mathcal{L}^{8}_{+}$	$\mathcal{L}^{9}_{+} \oplus \mathcal{L}^{3}_{+} \oplus \mathbb{R}^{5}_{+}$
9	$\mathcal{H}^{3}_{+}(\mathbb{C}) \oplus \mathcal{L}^{9}_{+}$	$\mathcal{L}^{10}_{+} \oplus \mathbb{R}^{8}_{+}$
10	$\mathcal{H}^{3}_{+}(\mathbb{C}) \oplus \mathcal{L}^{10}_{+}$	$\mathcal{L}^{9}_{+} \oplus \mathcal{L}^{7}_{+} \oplus \mathcal{L}^{3}_{+}$

n	K	$\sim$
15	$\mathcal{H}^3_+(\mathbb{C}) \oplus \mathcal{L}^{15}_+$	$\mathcal{L}^{14}_+ \oplus \mathcal{L}^{8}_+ \oplus \mathbb{R}^2_+$
18	$\mathcal{H}^3_+(\mathbb{C}) \oplus \mathcal{L}^{18}_+$	$\mathcal{L}^{14}_+ \oplus \mathcal{L}^{13}_+$
21	$\mathcal{H}^3_+(\mathbb{C}) \oplus \mathcal{L}^{21}_+$	$\mathcal{L}^{19}_+ \oplus \mathcal{L}^{11}_+$
22	$\mathcal{H}^3_+(\mathbb{C}) \oplus \mathcal{L}^{22}_+$	$\mathcal{L}^{21}_+ \oplus \mathcal{L}^{9}_+ \oplus \mathbb{R}^1_+$
30	$\mathcal{H}^3_+(\mathbb{C}) \oplus \mathcal{L}^{30}_+$	$\mathcal{L}^{29}_+ \oplus \mathcal{L}^{10}_+$

Conjecture.

$\mathcal{H}^3_+(\mathbb{C}) \oplus \mathcal{L}^n_+$ has symmetric similacra if and only if $n \in \{2, 3, \ldots, 10, 15, 18, 21, 22, 30\}$.

Conjecture.

If K is a symmetric cone with an $\mathcal{H}^3_+(\mathbb{C})$ factor and no symmetric similacra, then

$$K \cong \mathcal{H}^3_+(\mathbb{C}) \oplus \mathcal{L}^{n_1}_+ \oplus \mathcal{L}^{n_2}_+ \oplus \cdots \oplus \mathcal{L}^{n_k}_+$$

where $k \in \mathbb{N}$ and $n_i \notin \{2, 3, \ldots, 10, 15, 18, 21, 22, 30\}$.

Appendix: the table of ranks

K	$\dim(K)$	$\beta(K)$	n
$\mathbb{R}^n_+$	n	n	≥ 0
$\mathcal{L}^n_+$	n	$\frac{n^2-n+2}{2}$	≥ 1
$\mathcal{H}^n_+(\mathbb{R})$	$\frac{n^2+n}{2}$	n^2	≥ 2
$\mathcal{H}^n_+(\mathbb{C})$	n^2	$2n^2-1$	≥ 2
$\mathcal{H}^n_+(\mathbb{H})$	$2n^2-n$	$4n^2$	≥ 2
$\mathcal{H}^3_+(\mathbb{O})$	27	79	≥ 2

Where do the restrictions on n come from? Without them, the formulas can give wrong answers:

$$\begin{aligned} \beta\left(\mathcal{L}^{0}_{+}\right) &\neq \left(0^2 - 0 + 2\right)/2 \\ \beta\left(\mathcal{H}^{0}_{+}\left(\mathbb{C}\right)\right) &\neq 2\left(0^2\right) - 1 \\ \beta\left(\mathcal{H}^{1}_{+}\left(\mathbb{H}\right)\right) &\neq 4\left(1^2\right). \end{aligned}$$

$\mathcal{L}^{0}_{+} = \mathcal{H}^{0}_{+}\left(\mathbb{C}\right)$ is $\{0\}$, which has Lypaunov rank zero.

$\mathcal{H}^{1}_{+}\left(\mathbb{H}\right)$ is $\mathbb{R}^{1}_{+}$, which has Lyapunov rank one.

The Gowda/Tao formulas come from this table[5]:

V	K	$\mathfrak{g}$	$\mathfrak{k}$	$\dim(V)$	$\operatorname{rank}(V)$	d
$\mathcal{L}^n$	$\mathcal{L}^n_+$	$\mathfrak{o}(1, n-1) \oplus \mathbb{R}$	$\mathfrak{o}(n-1)$	n	2	$n-2$
$\mathcal{H}^n(\mathbb{R})$	$\mathcal{H}^n_+(\mathbb{R})$	$\mathfrak{sl}(n, \mathbb{R}) \oplus \mathbb{R}$	$\mathfrak{o}(n)$	$\frac{1}{2}n(n+1)$	n	1
$\mathcal{H}^n(\mathbb{C})$	$\mathcal{H}^n_+(\mathbb{C})$	$\mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{R}$	$\mathfrak{su}(n)$	n^2	n	2
$\mathcal{H}^n(\mathbb{H})$	$\mathcal{H}^n_+(\mathbb{H})$	$\mathfrak{sl}(n, \mathbb{H}) \oplus \mathbb{R}$	$\mathfrak{su}(n, \mathbb{H})$	$n(2n-1)$	n	4
$\mathcal{H}^3(\mathbb{O})$	$\mathcal{H}^3_+(\mathbb{O})$	$\mathfrak{e}_6(-26) \oplus \mathbb{R}$	$\mathfrak{f}_4$	27	3	8

5 **Analysis on symmetric cones**. Faraut and Korányi (1994), page 97.

The symbols $\mathfrak{g}$ and $\mathfrak{k}$ are defined 91 pages earlier...

$$\mathfrak{g} \coloneqq \operatorname{Lie}\left(\operatorname{Aut}\left(K\right)\right)$$
$$\mathfrak{k} \coloneqq \operatorname{Lie}\left(\operatorname{Aut}\left(K\right)_{\mathbf{1}_V}\right)$$

and it is known that

$$\operatorname{Aut}\left(K\right)_{\mathbf{1}_V} = \operatorname{JAut}\left(V\right)$$

and

$$\operatorname{Lie}\left(\operatorname{JAut}\left(V\right)\right) = \operatorname{Der}\left(V\right),$$

So $\mathfrak{g} = \mathbf{LL}(K)$ and $\mathfrak{k} = \mathrm{Der}(V)$:

V	K	$\mathbf{LL}(K)$	$\mathrm{Der}(V)$	$\dim(V)$	$\mathrm{rank}(V)$	d
$\mathcal{L}^n$	$\mathcal{L}^n_+$	$\mathfrak{o}(1, n-1) \oplus \mathbb{R}$	$\mathfrak{o}(n-1)$	n	2	$n-2$
$\mathcal{H}^n(\mathbb{R})$	$\mathcal{H}^n_+(\mathbb{R})$	$\mathfrak{sl}(n, \mathbb{R}) \oplus \mathbb{R}$	$\mathfrak{o}(n)$	$\frac{1}{2}n(n+1)$	n	1
$\mathcal{H}^n(\mathbb{C})$	$\mathcal{H}^n_+(\mathbb{C})$	$\mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{R}$	$\mathfrak{su}(n)$	n^2	n	2
$\mathcal{H}^n(\mathbb{H})$	$\mathcal{H}^n_+(\mathbb{H})$	$\mathfrak{sl}(n, \mathbb{H}) \oplus \mathbb{R}$	$\mathfrak{su}(n, \mathbb{H})$	$n(2n-1)$	n	4
$\mathcal{H}^3(\mathbb{O})$	$\mathcal{H}^3_+(\mathbb{O})$	$\mathfrak{e}_6(-26) \oplus \mathbb{R}$	$\mathfrak{f}_4$	27	3	8

This is wrong in even **more** $n = 0$ cases…so where did the entries (Lie algebras) come from? To the surprise of no one, F&K give no indication.

We can guess however:

- Lie algebras are finite-dimensional vector spaces
- All algebras of the same dimension are equivalent
- Did they just make up a Lie group/algebra of the desired size?

If so, how would they have known the dimensions?

Using a theorem in differential geometry:

$$\beta(K) = \dim(\operatorname{Der}(V)) + \dim(V)$$

As a result, $\dim(\operatorname{Der}(V))$ is all that is needed to fill in the table.

And $\dim(\operatorname{Der}(V))$ can be found in the book by Braun and Koecher[6]

[6] **Jordan-Algebren**. Hel Braun and Max Koecher (1966).

Satz (IX.9.3).

Es sei $\mathfrak{A}$ eine zentral-einfache Jordan-Algebra vom Grad r über $\mathbb{F}$. Ist dann die Charakteristik von $\mathbb{F}$ null, so ist die Dimension der Derivations-Algebra $\operatorname{Der}(\mathfrak{A})$ gleich $\frac{dr(\dim(\mathfrak{A})-1)}{4+(r-2)d}$.

Theorem (translated). If V is a (nontrivial) simple EJA, then

$$\dim(\operatorname{Der}(V)) = \frac{d \operatorname{rank}(V)(\dim(V) - 1)}{4 + (\operatorname{rank}(V) - 2)d}.$$

Do these dimensions agree with the Lie algebra listed for $\mathfrak{k} = \operatorname{Der}(V)$ in the table? Yes!

V	$\operatorname{Der}(V)$	$\dim(\operatorname{Der}(V))$	d
$\mathcal{L}^n$	$\mathfrak{o}(n-1)$	$\frac{1}{2}(n-1)(n-2)$	$n-2$
$\mathcal{H}^n(\mathbb{R})$	$\mathfrak{o}(n)$	$\frac{1}{2}n(n-1)$	1
$\mathcal{H}^n(\mathbb{C})$	$\mathfrak{su}(n)$	n^2-1	2
$\mathcal{H}^n(\mathbb{H})$	$\mathfrak{su}(n,\mathbb{H})$	$n(2n+1)$	4
$\mathcal{H}^3(\mathbb{O})$	$\mathfrak{f}_4$	52	8

And the entries for $\mathbf{LL}(K)$ have dimension $\dim(V) + \dim(\operatorname{Der}(V))$:

V	K	$\mathbf{LL}(K)$	$\dim(\operatorname{Der}(V)) + \dim(V)$	d
$\mathcal{L}^n$	$\mathcal{L}^n_+$	$\mathfrak{o}(1, n-1) \oplus \mathbb{R}$	$\frac{1}{2}\left(n^2 + n - 2\right)$	$n - 2$
$\mathcal{H}^n(\mathbb{R})$	$\mathcal{H}^n_+(\mathbb{R})$	$\mathfrak{sl}(n, \mathbb{R}) \oplus \mathbb{R}$	n^2	1
$\mathcal{H}^n(\mathbb{C})$	$\mathcal{H}^n_+(\mathbb{C})$	$\mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{R}$	$2n^2 - 1$	2
$\mathcal{H}^n(\mathbb{H})$	$\mathcal{H}^n_+(\mathbb{H})$	$\mathfrak{sl}(n, \mathbb{H}) \oplus \mathbb{R}$	$4n^2$	4
$\mathcal{H}^3(\mathbb{O})$	$\mathcal{H}^3_+(\mathbb{O})$	$\mathfrak{e}_6(-26) \oplus \mathbb{R}$	79	8

(for $n > 1$, at least)

Conclusion.

The Lie algebras in the F&K table have no meaning beyond having the “correct” dimensions.

Why, then, are they wrong for $\mathcal{L}_{+}^{0}$, $\mathcal{H}_{+}^{0}(\mathbb{C})$, and $\mathcal{H}_{+}^{1}(\mathbb{H})$?

For $\mathcal{L}_{+}^{0} = \mathcal{H}_{+}^{0}(\mathbb{C}) = \{0\}$, the dimension formula simply doesn’t apply (divide by zero).

Otherwise the dimensions are correct for the given d…it’s d that is wrong in some cases.

Recall (Satz IX.3.3):

Theorem. If V is a simple EJA, then

$$\dim\left(\operatorname{Der}\left(V\right)\right) = \frac{d \operatorname{rank}\left(V\right)\left(\dim\left(V\right) - 1\right)}{4 + \left(\operatorname{rank}\left(V\right) - 2\right) d}.$$

- d is the dimension of a subspace in the Peirce decomposition
- It must be nonnegative
- It depends on which simple EJA you're in

…but it also depends on the size n:

- $\mathcal{H}^2_+(\mathbb{F})$ should use the value of $d = \dim(V) - 2$ from the Lorentz cone, and **not** $d = 1, 2, 4$
- $\mathcal{L}^0_+ = \mathcal{H}^0_+(\mathbb{F})$ are special cases with $d = 0$
- $\mathcal{L}^1_+ = \mathcal{H}^1_+(\mathbb{F})$ are special cases with $d = 0$

In short, the d in the table are wrong for $n \leq 2$:

d	$n-2$	1	2	4	8
V	$\mathcal{L}^n$	$\mathcal{H}^n(\mathbb{R})$	$\mathcal{H}^n(\mathbb{C})$	$\mathcal{H}^n(\mathbb{H})$	$\mathcal{H}^n(\mathbb{O})$

But actually…

- By some miracle, the correct $d = n - 2$ happens to agree with $d = 1, 2, 4$ for

$$\mathcal{H}^2_+(\mathbb{R}^n) \cong \mathcal{L}^3_+ \quad (n = 3)$$
$$\mathcal{H}^2_+(\mathbb{C}^n) \cong \mathcal{L}^4_+ \quad (n = 4)$$
$$\mathcal{H}^2_+(\mathcal{H}^n) \cong \mathcal{L}^6_+ \quad (n = 6)$$

- The value of d for $\mathcal{L}^1_+$ turns out to be irrelevant, because the numerator in the Braun/Koecher formula is zero.

Conclusion.

- The F&K table is correct for $n \geq 2$.
- The F&K table is correct for $n = 1$ if you use the $\mathcal{L}^1_+$ row.
- Don't use the F&K table for $n = 0$.

K	$\dim(K)$	$\beta(K)$	n
$\mathbb{R}^n_+$	n	n	≥ 0
$\mathcal{L}^n_+$	n	$\frac{n^2-n+2}{2}$	≥ 1
$\mathcal{H}^n_+(\mathbb{R})$	$\frac{n^2+n}{2}$	n^2	≥ 2
$\mathcal{H}^n_+(\mathbb{C})$	n^2	$2n^2-1$	≥ 2
$\mathcal{H}^n_+(\mathbb{H})$	$2n^2-n$	$4n^2$	≥ 2
$\mathcal{H}^3_+(\mathbb{O})$	27	79	≥ 2

The end