THE STRUCTURE OF THE GROUP OF AUTOMORPHISMS OF A HOMOGENEOUS CONVEX CONE[1)]

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CONTENTS

Introduction

An *automorphism* of a convex cone V in an n-dimensional real space R is a nondegenerate linear transformation of the space R which leaves the cone V invariant. We denote the *group of all automorphisms* of the cone V by $\mathfrak{G}(V)$.

The cone V is said to be *homogeneous* if the group $\mathfrak{G}(V)$ acts on it transitively. In this case the group $\mathfrak{G}(V)$ is a subgroup of finite index in some algebraic linear group [2]. Let G be the Lie algebra of the group $\mathfrak{G}(V)$. We shall refer to the elements of G as the *derivations* of the cone V. Most of the results of this paper refer to derivations and not to automorphisms.

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1) The main results of this paper were presented at the meeting of the Moscow Mathematical Society on the 10th of April 1962.

An important special case of the set of homogeneous convex cones is the set of *selfadjoint homogeneous cones* (Chapter II). In this case, and only in this case, the algebra of derivations is reducible. A description of the algebras of derivations of selfadjoint cones is given in Chapter II.

If V is an arbitrary homogeneous cone, then its algebra of derivations G admits, as does every algebraic linear Lie algebra, a decomposition of the form $G = N + A + S$, where N is a maximal ideal consisting of nilpotent endomorphisms, A is an Abelian algebraic subalgebra consisting of semisimple endomorphisms and S is a semisimple subalgebra commuting with A ([7], Proposition 5, §4, Chapter V). We denote the sums of all the noncompact and compact ideals of the algebra S by S_1 and S_0, respectively, and the sets of all the elements in A which have real and pure imaginary eigenvalues by A_1 and A_0, respectively. Then

$$G = N + G^c + G_0,$$

where

$$G^c = A_1 + S_1, \quad G_0 = A_0 + S_0.$$

In this notation most of the results of this paper can be formulated as follows:

1°. The weights of the algebra A_1 over the space R are of the form $(\psi_\alpha + \psi_\beta)/2$, where $1 \le \alpha \le \beta \le \mu$ and $\{\psi_1, \cdots, \psi_\mu\}$ is some basis in the space of linear functionals on A_1. The corresponding weight subspaces will be denoted by $R_{\alpha\beta}$. They are clearly invariant with respect to $G^c + G_0$. Some of the spaces $R_{\alpha\beta}$ may be equal to 0 but the spaces $R_{\alpha\alpha}$ are never zero.

2°. The intersection of the space $R^c = \Sigma R_{\alpha\alpha}$ with the cone V is a homogeneous selfadjoint cone V^c, called the *kernel* of the cone V. It splits into the direct sum of cones V_α lying in the subspaces $R_{\alpha\alpha}$.

3°. The restriction of the algebra G^c to R^c is its isomorphic image in the algebra of derivations of the cone V^c.

4°. The transformations in G_0 map R^c into 0.

5°. The weights of the adjoint representation of the algebra A_1 over the ideal N are of the form $(\psi_\alpha - \psi_\beta)/2$, where $1 \le \alpha < \beta \le \mu$. The corresponding weight subspaces will be denoted by $N_{\alpha\beta}$. They are invariant with respect to $G^c + G_0$.

6°. If $v \in V_\beta$ and $\alpha < \beta$ the mapping $A \to Av$, $A \in N_{\alpha\beta}$, is an isomorphism of the space $N_{\alpha\beta}$ onto the space $R_{\alpha\beta}$.

The results 1° – 6° follow easily from the results of Chapter IV, where we describe the action of the algebra G in the space R in terms of the *generalized* *matrix calculus* constructed in [2]. In this calculus a homogeneous convex cone appears as the cone of "positive-definite" Hermitian generalized matrices. A precise definition of the matrix calculus is given in Chapter I. In Chapter III we construct a modification of it which is adapted to the description of the derivations of a cone.

The action of the algebra $G^c + N$ in the space R is described in Chapter IV in explicit detail. The results for the algebra G_0 are not so complete. In particular, finding the algebra G_0 is an independent problem, the solution of which requires quite different methods. It is clear from examples that the semisimple part S_0 of the algebra G_0 may be isomorphic to any compact semisimple Lie algebra. Moreover, the representations of the algebra S_0 in the space R may contain any of its irreducible representations.

CHAPTER I
Matrix Calculus

§1. Definition of a T-algebra

In [2] we introduced the so-called *T-algebras* which are a natural means of discussing and studying homogeneous convex cones. For the convenience of the reader we shall now give the relevant definitions.

A *matrix algebra with involution* is an algebra $\mathfrak{A}$ which is bigraded by the subspaces $\mathfrak{A}_{ij}$ $(i, j = 1, \cdots, m)$ and provided with an involutive antiautomorphisms* in such a way that

1) $\mathfrak{A}_{ij}\mathfrak{A}_{lk}\begin{cases} \subset \mathfrak{A}_{ik} & \text{when } j = l, \\ = 0 & \text{when } j \neq l; \end{cases}$

2) $\mathfrak{A}^*_{ij} = \mathfrak{A}_{ji}$.

The number m is called the *rank* of the matrix algebra $\mathfrak{A}$.

The elements of the matrix algebra are conveniently represented as matrices of the form (a_{ij}), where $a_{ij} \in \mathfrak{A}_{ij}$. The matrix (a_{ij}) is said to be *Hermitian* if $a^*_{ij} = a_{ji}$. *Skew-Hermitian* matrices may be defined in the same way.

We shall adopt the convention that the symbols a_{ij}, x_{ij}, etc., will always denote arbitrary elements of the subspace $\mathfrak{A}_{ij}$ of the matrix algebra $\mathfrak{A}$.

A matrix algebra with involution is said to be a *T-algebra* if the following conditions (*axioms*) are satisfied:

1) for any i the subalgebra $\mathfrak{A}_{ii}$ is one-dimensional and admits an isomorphic mapping ρ onto the algebra of real numbers;

2) $a_{ii}b_{ij} = \rho(a_{ii})\, b_{ij}$;

3) there exist numbers $n_i > 0$ such that

$$n_i\rho(a_{ij}b_{ji}) = n_j\rho(b_{ji}a_{ij});$$

4) $\rho(a_{ij}a_{ij}^*) > 0$ when $a_{ij} \neq 0$;

5) $a_{ij}(b_{jk}c_{ki}) = (a_{ij}b_{jk})c_{ki}$;

6) $a_{ij}(b_{jk}c_{kl}) = (a_{ij}b_{jk})c_{kl}$ when $i < j < k$ and $j < l$;

7) $a_{ij}(b_{jk}b_{jk}^*) = (a_{ij}b_{jk})b_{jk}^*$ when $i < j < k$.

From axioms 2), 5) and 7) we easily deduce the important relation

$$\rho((a_{ij}b_{jk})(a_{ij}b_{jk})^*) = \rho(a_{ij}a_{ij}^*)\rho(b_{jk}b_{jk}^*) \text{ when } i \leq j \leq k. \tag{1}$$

Let $\mathfrak{X}$ denote the space of Hermitian matrices in the T-algebra $\mathfrak{A}$. The set $V(\mathfrak{A})$ of matrices which are expressible in the form tt^*, where t is an upper triangular matrix with positive elements on the main diagonal, is a homogeneous convex cone in the space $\mathfrak{X}$. The transformations

$$D_t\colon\ x \longrightarrow tx + xt^* \quad (x \in \mathfrak{X}),$$

where t is an arbitrary upper triangular matrix, are derivations of the cone $V(\mathfrak{A})$. They form a Lie algebra. The Lie group generated by them acts simply-transitively in the cone $V(\mathfrak{A})$.

For every homogeneous convex cone V there exists a T-algebra $\mathfrak{A}$ such that the cones V and $V(\mathfrak{A})$ are isomorphic. The positive numbers n_i appearing in axiom 3) for a T-algebra may be specified arbitrarily but beyond that the algebra $\mathfrak{A}$ is uniquely determined.

In what follows we shall consider only canonical T-algebras, for which

$$n_i = 1 + \frac{1}{2}\sum_{s \neq i} \dim \mathfrak{A}_{is}. \tag{2}$$

This choice of the numbers n_i is connected with the invariant measure in the cone $V(\mathfrak{A})$.

§2. Canonical Riemannian geometry of a cone

In every convex cone V we may define the function

$$\phi(x) = \int_{y \in V'} e^{-(x,y)}\, dy,$$

where the integral is taken over the adjoint cone V' and dy is the Euclidean measure. The function ϕ has the following properties [3, 2]:

1) for every automorphism A of the cone V

$$\phi(Ax) = \frac{\phi(x)}{\det A};$$

2) in an affine coordinate system the quadratic form $d^2\log\phi$ is positive

definite at any point of the cone V;

3) the function ϕ grows without limit on approach to the boundary of the cone V.

The property 1) shows that the measure $\phi(x)\,dx$ is invariant under automorphisms of the cone V. If the quadratic form $d^2\log\phi$ is considered only in affine coordinate systems then it defines an *invariant Riemannian metric* in the cone V. Let Γ be an object of the linear connectivity associated with this metric. In affine coordinate systems the object Γ may be regarded as a tensor which is twice covariant and once contravariant.

Let x_0 be some point of the cone V. The *connectedness algebra* of the cone V at the point x_0 is the algebraic structure in the tangent space to the cone V at the point x_0 defined by the formula

$$(a \square b)^i = -\Gamma^i_{jk}(x_0)\, a^j b^k$$

(the square denotes the operation of multiplication in the connectedness algebra). If the cone V lies in a linear space R then the tangent space to the cone V at any point on it may be identified with the space R, by means of a parallel translation. Therefore the structure of the connectedness algebra may also be discussed in the space R itself.

The connectedness algebras of the cone V at its different points form a "field of algebras" which is invariant under the automorphisms of the cone V. In particular, the connectedness algebra at some point $x_0 \in V$ is invariant under the automorphisms which leave this point fixed. This fact will be used in what follows.

We now present the formulas which describe the Riemannian metric and the connectedness algebra of the homogeneous convex cone V in terms of the canonical T-algebra $\mathfrak{A}$ corresponding to it (cf. §1).

For any matrix $a = (a_{ij}) \in \mathfrak{A}$ we put

$$\operatorname{Sp} a = \sum n_i \rho(a_{ii}) \tag{3}$$

The unit matrix e belongs to the cone V and at the point e the Riemannian metric is given by the formula

$$(x, y) = \operatorname{Sp} xy,$$

and the connectedness algebra by the formula

$$x \square y = \frac{1}{2}(xy + yx),$$

where x and y are arbitrary Hermitian matrices, considered as elements of the tangent space to the cone V at the point e.

§3. Associativity relations in a T-algebra

The axioms 5)–7) for a T-algebra are associativity relations. Other such relations can be derived from them.

Proposition 1. *In every T-algebra $\mathfrak{A}$*

$$a_{ij}(b_{jk}c_{kl}) = (a_{ij}b_{jk})c_{kl}, \tag{4}$$

if $i \neq k$, $j \neq l$ and the pair of numbers i, k does not differ (strictly) from the pair j, l by any point on the real axis. Moreover

$$a_{ij}(b_{jk}b^*_{jk}) = (a_{ij}b_{jk})b^*_{jk} \tag{5}$$

and

$$a^*_{ij}(a_{ij}b_{jk}) = (a^*_{ij}a_{ij})b_{jk}, \tag{6}$$

if j lies between i and k.

To prove this we introduce the following scalar product in the algebra $\mathfrak{A}$:

$$(a, b) = \mathrm{Sp}\, ab^*$$

(cf. (3)). It follows from axiom 4) for a T-algebra that $(a, a) > 0$ for $a \neq 0$. Axiom 3) is equivalent to the result that

$$(a^*, b^*) = (a, b)$$

for any $a, b \in \mathfrak{A}$. Further, axiom 5) implies that

$$(ab, c) = (a, cb^*) = (b, a^*c)$$

for any $a, b, c \in \mathfrak{A}$.

We now show how to prove the formula (4), for example, in the case when $j < k < l < i$. We have that

$$(a_{ij}(b_{jk}c_{kl}), x_{il}) = (b_{jk}c_{kl}, a^*_{ij}x_{il}) = ((b_{jk}c_{kl})x^*_{il}, a^*_{ij})$$
$$= (b_{jk}(c_{kl}x^*_{il}), a^*_{ij}) = (x^*_{il}, c^*_{kl}(b^*_{jk}a^*_{ij})) = ((a_{ij}b_{jk})c_{kl}, x_{il}),$$

which implies that $a_{ij}(b_{jk}c_{kl}) = (a_{ij}b_{jk})c_{kl}$.

The remaining associativity relations are proved in the same way. Some of them are obtained from others by use of the involution.

§4. The kernel of a T-algebra

In this section we shall construct something like an "*associative center*" of a T-algebra. It will play an important part in the solution of our problem.

Let $\mathfrak{A}$ be a T-algebra of rank m and M the set of indices $1, 2, \cdots, m$. For brevity we put

$$n_{ij} = \dim \mathfrak{A}_{ij}.$$

By virtue of the axioms for a T-algebra, $n_{ii} = 1$ and $n_{ij} = n_{ji}$ for any $i, j \in M$.

The permutation $i \to \tilde{i}$ of the set M is said to be *admissible* if $i < j$, $\tilde{i} > \tilde{j}$ implies that $n_{ij} = 0$. Every admissible permutation defines an "*inessential*" *change in the gradation* of the algebra $\mathfrak{A}$, consisting in the renaming of the space $\mathfrak{A}_{ij}$ as $\mathfrak{A}_{\tilde{i}\tilde{j}}$. The T-algebra obtained by means of the new gradation is, by definition, isomorphic to the original T-algebra.

Let $\mathfrak{R}$ be the set of all equivalence relations in the set M. We introduce a partial ordering in $\mathfrak{R}$ by putting $R_1 \leq R_2$ ($R_1, R_2 \in \mathfrak{R}$) if $i \equiv j \pmod{R_1}$ implies that $i \equiv j \pmod{R_2}$.

We first consider the equivalence relations $\tilde{R} \in \mathfrak{R}$ defined by the following condition: $i \equiv j \pmod{\tilde{\mathfrak{R}}}$ if $n_{ij} \neq 0$ and $n_{is} = n_{js}$ for all $s \neq i, j$. (It is easy to see that this relation is transitive.)

Let $\tilde{\mathfrak{R}}$ denote the set of equivalence relations $R \in \mathfrak{R}$ satisfying the following two conditions:

(A) $R \leq \tilde{R}$;

(B) if $i < j < k$ and $i \equiv k \pmod{R}$ then either $n_{ij} = n_{jk} = 0$ or $i \equiv j \equiv k \pmod{R}$.

It turns out that there is a maximal element $\bar{R}$ in the set $\tilde{\mathfrak{R}}$. The relation $\bar{R}$ can be defined by the following inductive construction:

1) $i \equiv i \pmod{\bar{R}}$ for all $i \in M$;

2) if we have already determined whether all the $i, j \in M$ such that $|i - j| < p$ are comparable modulo $\bar{R}$ or not, then for $|i - j| = p$ we put $i \equiv j \pmod{\bar{R}}$ if $i \equiv j \pmod{\tilde{R}}$ and for all s lying between i and j either $n_{is} = n_{sj} = 0$ or $i \equiv s \pmod{\bar{R}}$ and $s \equiv j \pmod{\bar{R}}$.

The relation $\bar{R}$ defines a partition of the set M into disjoint equivalence classes M_α ($\alpha = 1, \cdots, \mu$). Let $i_1^{(\alpha)}, \cdots, i_{m_\alpha}^{(\alpha)}$ be elements of the set M_α, arranged in increasing order. If necessary we renumber the sets M_α so that we have the inequalities $i_1^{(1)} < i_1^{(2)} < \cdots < i_1^{(\mu)}$ and we arrange the elements of the set M as follows:

$$i_1^{(1)}, \cdots, i_{m_1}^{(1)}, i_1^{(2)}, \cdots, i_1^{(\mu)}, \cdots, i_{m_\mu}^{(\mu)}.$$

Let $\tilde{i}$ be the ordinal of the number $i \in M$ in this arrangement. We shall show that the permutation $i \to \tilde{i}$ is admissible.

Suppose that $i < j$ but that $\tilde{i} > \tilde{j}$. If $i \in M_\alpha$, $j \in M_\beta$ then $\alpha \geq \beta$. In fact $\alpha > \beta$ since otherwise i and j would belong to the same set M_α and their relative position would be unaltered. Since

$$i_1^{(\beta)} < i_1^{(\alpha)} \leq i < j$$

and $i_1^{(\beta)} \equiv j \pmod{\bar{R}}$ it follows that either $n_{ij} = 0$ or $i \equiv j \pmod{\bar{R}}$. The latter is clearly impossible and so $n_{ij} = 0$ which is what we had to prove.

Thus, by means of an inessential change in the gradation we can arrange for the relation $\bar{R}$ to satisfy the following condition, which is stronger than (B):

(B′) if $i < j < k$ and $i \equiv k \pmod{\bar{R}}$, then $i \equiv j \equiv k \pmod{\bar{R}}$.

For every $i \in M$ we define $\alpha(i)$ so that $i \in M_{\alpha(i)}$. It follows from (B′) that if $i < j$ then $\alpha(i) \leq \alpha(j)$. In what follows we shall assume that this condition always holds.

The property (A) of the relation $\bar{R}$ implies that when i and j run through the set M_α, remaining distinct, the number n_{ij} remains constant; we shall denote this value by $\nu_{\alpha\alpha}$. If i runs through the set M_α and j the set M_β $(\beta \neq \alpha)$, then n_{ij} also has a constant value which we shall denote by $\nu_{\alpha\beta}$. Thus

$$n_{ij} = \begin{cases} \nu_{\alpha(i)\alpha(j)}, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases}$$

The subspace

$$\mathfrak{A}^c = \sum_{i \equiv j \pmod{\bar{R}}} \mathfrak{A}_{ij}$$

will be called the *kernel* of the T-algebra $\mathfrak{A}$.

Clearly $\mathfrak{A}^c$ is a subalgebra which is invariant under involution. We provide it with a gradation by putting

$$\mathfrak{A}_{ij}^c = \mathfrak{A}^c \cap \mathfrak{A}_{ij} = \begin{cases} \mathfrak{A}_{ij}, & \text{if } i \equiv j \pmod{\bar{R}}. \\ 0, & \text{if } i \not\equiv j \pmod{\bar{R}}. \end{cases}$$

It is easy to see that, with this gradation, $\mathfrak{A}^c$ becomes a T-algebra.

We observe that if $i \equiv j \pmod{\bar{R}}$ then $n_i = n_j$ (cf. (2)).

Proposition 2. *The relations* (5) *and* (6) *(with* $i \neq k$*) and the relation* (4) *(with* $i \neq k$ *and* $j \neq l$*) are automatically satisfied if at least one of the elements appearing in them belongs to the kernel* $\mathfrak{A}^c$ *of the T-algebra* $\mathfrak{A}$.

Proof. Let $i < j < k$ and $a_{ij} \in \mathfrak{A}_{ij}$. We consider the mapping

$$x_{jk} \to a_{ij} x_{jk} \tag{7}$$

of the space $\mathfrak{A}_{jk}$ into the space $\mathfrak{A}_{ik}$. This mapping is invertible. In fact

$$a_{ij}^*(a_{ij} x_{jk}) = (a_{ij}^* a_{ij}) x_{jk} = \rho(a_{ij}^* a_{ij}) x_{jk}.$$

We now assume that $i \equiv j \pmod{\bar{R}}$. Then the dimensions of the spaces $\mathfrak{A}_{jk}$ and $\mathfrak{A}_{ik}$ are equal and the image of the space $\mathfrak{A}_{jk}$ under the mapping (7) coincides with $\mathfrak{A}_{ik}$. This is the basis for the proof of Proposition 2.

We confine ourselves here to the proof of the fact that, when $i < j < k$ and $i \equiv j \pmod{\overline{R}}$,

$$a^*_{ij}(b_{ik}b^*_{ik}) = (a^*_{ij}b_{ik})b^*_{ik},$$

i.e. of a relation of the form (5) which, in general, does not hold if $i \not\equiv j \pmod{\overline{R}}$.

Expressing b_{ik} in the form $b_{ik} = a_{ij}u_{jk}$, we find that

$$(a^*_{ij}(b_{ik}b^*_{ik}) = \rho(b_{ik}b^*_{ik})\,a^*_{ij} = \rho(a_{ij}a^*_{ij})\,\rho(u_{jk}u^*_{jk})\,a^*_{ij} = \rho(a^*_{ij}a_{ij})\,((u_{jk}u^*_{jk})\,a^*_{ij})$$
$$= \rho(a^*_{ij}a_{ij})\,(u_{jk}b^*_{ik}) = ((a^*_{ij}a_{ij})\,u_{jk})b^*_{ik} = (a^*_{ij}b_{ik})b^*_{ik}.$$

(Here we have used the relation (6), Proposition 1, axiom 3) for a T-algebra and the equation $n_i = n_j$.)

The other associativity relations involving elements of $\mathfrak{A}^c$ are proved in the same way.

CHAPTER II

Homogeneous selfadjoint cones

§1. Classification and the connection with Jordan algebras

An open convex cone V in a linear space R is said to be *selfadjoint* if the following two conditions are satisfied in an appropriate Euclidean metric:

1) $(x, y) > 0$ for all $x, y \in V$;

2) if $x \in R$ is a vector such that $(x, y) \geq 0$ for all $y \in \overline{V}$ then $x \in \overline{V}$.

If the nondegenerate linear transformation A of the space R leaves the selfadjoint cone V invariant then so does the adjoint transformation A^*. In other words *if the group of all automorphisms of a selfadjoint cone contains a transformation then it also contains its adjoint.* This implies that *it is completely reducible.* It will become clear from what follows that among the homogeneous cones it is only the selfadjoint ones that have a completely reducible group of automorphisms.

In [1, 4] and [5] a connection was established between the homogeneous selfadjoint cones and compact Jordan algebras and on this basis a complete classification of homogeneous selfadjoint cones was obtained. It turned out that each such cone splits into the direct sum of indecomposable homogeneous selfadjoint cones (having an irreducible group of automorphisms); the latter are of the following five types:

I. The cone of positive-definite symmetric nth order matrices.

II. The cone of positive-definite Hermitian nth order matrices.

III. The cone of positive-definite Hermitian quaternion nth order matrices.

IV. The cone of positive-definite Hermitian octavic third order matrices.[1)]

V. The spherical cone

$$x_0 > \sqrt{x_1^2 + \cdots + x_n^2}.$$

In the canonical Riemannian geometry (cf. Chapter I) the homogeneous selfadjoint cones are symmetric spaces of nonpositive curvature. In our terminology the Jordan algebra corresponding to such a cone is none other than its connectedness algebra.

Let V be a homogeneous selfadjoint cone corresponding to the Jordan algebra J. The operator of multiplication by the element a in the algebra J will be denoted by R_a. For any a this operator is symmetric in the Euclidean metric $(a, b) = \operatorname{sp} R_{ab}$. The cone V may be described as the set of $x \in J$ for which the operator R_x is positive-definite. In particular, the unit element e of the algebra J belongs to the cone V. The stationary subgroup $\mathfrak{H}$ of the point $e \in V$ in the group $\mathfrak{G}$ of all automorphisms of the cone V coincides with the group of automorphisms of the algebra J. Every element of the group $\mathfrak{G}$ is uniquely expressible in the form of the product of an automorphism of the algebra J and the transformation $\exp R_a$, $a \in J$, which is a parallel translation along the geodesic of the cone V passing through the element e.

Among the indecomposable homogeneous cones enumerated above a particular place is occupied by the cone of type IV. This will become completely clear in Chapter IV (Theorem 1). We shall refer to it as the *exceptional* cone. It is of dimension 27. The Jordan algebra corresponding to it is the only exceptional compact Jordan algebra. Its group of automorphisms is connected and is the direct product of a one-parameter similitude group and a real simple group of type E_6; the stationary subgroup is isomorphic to a simple compact group of type F_4 [8].

§2. The construction of T-algebras

In this section we shall construct the T-algebras (cf. the introduction) corresponding to indecomposable selfadjoint cones. We shall see that these T-algebras are characterized by the following property:

$$\dim \mathfrak{A}_{ij} = \text{const} \quad \text{when } i \neq j. \tag{1}$$

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1) The Hermitian octavic third order matrices form a Jordan algebra under the operation $a \square b = (ab + ba)/2$. By means of some automorphism from this algebra every Hermitian matrix a can be reduced to a diagonal form in which the real numbers appearing on the main diagonal are determined uniquely except for order [6] and are called the *eigenvalues* of the matrix a. The matrix a is said to be *positive-definite* if all its eigenvalues are positive.

We shall use A^ν ($\nu = 1, 2, 4, 8$) to denote the ν-dimensional division algebra over the real field, i.e. the algebra of real numbers, of complex numbers, of quaternions, of octaves (Cayley numbers). Suppose further that $\overline{\mathfrak{A}}_m^\nu$ ($\nu = 1, 2, 4$; m is any natural number; or $\nu = 8$ and $m = 3$) is an algebra of mth order matrices with elements in the algebra A^ν. We shall denote the multiplication of matrices in $\overline{\mathfrak{A}}_m^\nu$ by a dot.

The algebra $\overline{\mathfrak{A}}_m^\nu$ can be equipped with an involution (transposition and conjugation) in a natural way. For every matrix $a \in \overline{\mathfrak{A}}_m^\nu$ we denote by Pa the matrix obtained from a by replacement of the diagonal elements by their real parts. We consider the space

$$\mathfrak{A}_m^\nu = P\overline{\mathfrak{A}}_m^\nu.$$

We provide it with multiplication by means of the formula

$$ab = P(a \cdot b) \quad (a, b \in \mathfrak{A}_m^\nu).$$

Clearly, $\mathfrak{A}_m^\nu$ then becomes a matrix algebra with involution. It is not difficult to verify that $\mathfrak{A}_m^\nu$ is a T-algebra. In particular, when $\nu = 8$, $m = 3$ axiom 6) of a T-algebra becomes void and axiom 7) is satisfied because of the alternating property of the algebra of octaves.

The T-algebras so constructed correspond to indecomposable selfadjoint cones of the first four types. Indeed, for any triangular matrix $t \in \mathfrak{A}_m^\nu$

$$tt^* = P(t \cdot t^*) = t \cdot t^*;$$

on the other hand, it is well known that a symmetric (or Hermitian or Hermitian quaternion) matrix is positive-definite if and only if it can be expressed in the form $t \cdot t^*$, where t is a triangular matrix with positive elements on the main diagonal. The same holds for Hermitian octavic third order matrices.

Corresponding to the spherical cone in $(\nu + 2)$-dimensional space is the unique T-algebra of rank 2 for which $n_{12} = \nu$. We shall denote this T-algebra by $\mathfrak{A}_2^\nu$.

Proposition 3. *Every T-algebra $\mathfrak{A}$ having the property* (1) *is isomorphic to one of the T-algebras*

$$\mathfrak{A}_m^1\,(m \geqslant 1),\ \mathfrak{A}_m^2\,(m \geqslant 2),\ \mathfrak{A}_m^4\,(m \geqslant 3),\ \mathfrak{A}_3^8,\ \mathfrak{A}_2^\nu\ (\nu \geqslant 3).$$

Proof. If the rank of the algebra $\mathfrak{A}$ is two then it is isomorphic to the algebra $\mathfrak{A}_2^\nu$ for some ν. Therefore we shall assume that the rank of the algebra $\mathfrak{A}$ is $m \geq 3$.

The property (1) means that the algebra $\mathfrak{A}$ coincides with its kernel (cf. §4, Chapter I). Therefore the associativity relations given in Proposition 2 hold in $\mathfrak{A}$ for any values of the sympols appearing in them. This will be the basis of our proof.

In each of the subspaces $\mathfrak{A}_{i,i+1}$ $(i = 1, \cdots, m-1)$ we choose an arbitrary vector $e_{i,i+1}$ for which $\rho(e_{i,i+1}e^*_{i,i+1}) = 1$ and for $i < j$ we put

$$\left.\begin{aligned} e_{ij} &= e_{i,i+1}e_{i+1,i+2}\cdots e_{j-1,j} \\ e_{ji} &= e^*_{ij}. \end{aligned}\right\} \tag{2}$$

Let e_{ii} denote the element in $\mathfrak{A}_{ii}$ for which $\rho(e_{ii}) = 1$. If we use the associativity relations we can easily show that

$$e_{ij}e_{jk} = e_{ik} \tag{3}$$

for any i, j, k.

For $i \neq j$ and $k \neq 1$ we define the mapping θ^{ij}_{kl} of the space $\mathfrak{A}_{ij}$ onto the space $\mathfrak{A}_{kl}$ by putting

$$\theta^{ij}_{kl}(a_{ij}) = \begin{cases} e_{ki}(a_{ij}e_{jl}), & \text{if } i \neq l, \\ (e_{ki}a_{ij})e_{jl}, & \text{if } k \neq j. \end{cases}$$

(If $i \neq 1$ and also $k \neq j$ then $e_{ki}(a_{ij}e_{jl}) = (e_{ki}a_{ij})e_{jl}$.) Using the associativity relations and equations (2), (3) we easily establish that

$$\theta^{kl}_{pq}\theta^{ij}_{kl} = \theta^{ij}_{pq}.$$

Therefore we may choose a "standard" linear space A and for every pair i, j, where $i \neq j$, define an isomorphic mapping θ^{ij} of the space $\mathfrak{A}_{ij}$ onto A in such a way that

$$\theta^{kl}\theta^{ij}_{kl} = \theta^{ij}.$$

Here the elements $e_{ij} \in \mathfrak{A}_{ij}$ all go over into the same element $e \in A$, since

$$e_{kl} = \theta^{ij}_{kl}(e_{ij}).$$

We provide the space A with a Euclidean metric by means of the formula

$$(\theta^{1m}(a_{1m}),\ \theta^{1m}(b_{1m})) = \rho(a_{1m}b^*_{1m}).$$

Then, for any i, j

$$(\theta^{ij}(a_{ij}),\ \theta^{ij}(b_{ij})) = \rho(a_{ij}b^*_{ij}). \tag{4}$$

In fact, if $i < j$ then, if we use the formula (1) of Chapter I, we find that

$$(\theta^{ij}(a_{ij}),\ \theta^{ij}(b_{ij})) = (\theta^{1m}(e_{1i}a_{ij}e_{jm}),\ \theta^{1m}(e_{1i}b_{ij}e_{jm}))$$

$$= \rho((e_{1i}a_{ij}e_{jm})(e_{1i}b_{ij}e_{jm})^*) = \rho(e_{1i}e^*_{1i})\,\rho(a_{ij}b^*_{ij})\,\rho(e_{jm}e^*_{jm}) = \rho(a_{ij}b^*_{ij}),$$

which is what we had to show. We shall not examine the case $i > j$ since we shall not encounter it later.

We introduce a multiplication into the space A by putting

$$\theta^{ij}(a_{ij})\theta^{ik}(b_{jk}) = \theta^{ik}(a_{ij}b_{jk}) \tag{5}$$

for different i, j, k. It is easy to show that this definition is not inconsistent. Thus the space A becomes a ν-dimensional algebra over the real field, one equipped with a Euclidean metric. The element $\epsilon = \theta^{ij}(e_{ij})$ serves as the unit element of the algebra A. Clearly

$$(\epsilon, \epsilon) = 1. \tag{6}$$

The relation (5) and the formula (1) of Chapter I imply that for any elements α, $\beta \in A$

$$(\alpha\beta, \alpha\beta) = (\alpha, \alpha)(\beta, \beta). \tag{7}$$

By Hurwitz's theorem the algebra A is isomorphic to one of the algebras A^ν ($\nu = 1, 2, 4$, or 8) (cf. the definitions at the beginning of this section). The Euclidean metric with the properties (6) and (7) is unique.

The formulae (4) and (5) allow us to reconstruct the T-algebra $\mathfrak{A}$ from the algebra A and the number m. It is easy to check that it is isomorphic to the T-algebra $\mathfrak{A}_m^\nu$, where $\nu = \dim A$.

§3. Automorphisms

The connected component of the unit element of the group of automorphisms of the cone of positive-definite symmetric (or Hermitian or Hermitian quaternion) matrices consists of the transformations of the form

$$\pi(u)\colon\ x \longrightarrow u \cdot x \cdot u^*,$$

where u is a nondegenerate real (or complex, or quaternion) matrix.[1)] This assertion is equivalent to saying that the derivations of the corresponding cone are just the transformations of the form

$$D_a\colon\ x \longrightarrow a \cdot x + x \cdot a^*$$

where a is any matrix (depending on the cone: real, complex or quaternion).[2)]

Thus the automorphisms and derivations of indecomposable selfadjoint cones of the first three types are conveniently described in terms of the corresponding associative matrix algebras $\overline{\mathfrak{A}}_m^\nu$ ($\nu = 1, 2$, or 4). To represent the automorphisms and derivations of the spherical cone in the same way (and this will be very important later) we define the associative algebra $\overline{\mathfrak{A}}_2^\nu$ ($\nu \geq 3$) as an algebra of second order matrices with elements in the Clifford algebra $A^{(\kappa)}$, $\kappa = \nu - 1$, with generators $p_1, \cdots, p_\kappa$, satisfying the relations

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1) This does not hold for the full group of automorphisms. Namely, the transformation $x \to \bar{x}$ in the space of Hermitian complex matrices is an automorphism of the cone of positive-definite Hermitian matrices but is not expressible in the above form. Incidentally, this is the only exception.

2) The above assertions are easily deduced from the general description of the group of automorphisms of a homogeneous selfadjoint cone, given in §1 of this chapter.

$$p_i^2 = -1,\ p_i p_j + p_j p_i = 0 \quad (i \neq j).$$

Multiplication in the algebra $\bar{\mathfrak{A}}_2^\nu$ will be denoted by a dot.

We denote the space of homogeneous elements of degree s in the algebra $A^{(\kappa)}$ by $A_s^{(\kappa)}$. In particular, $A_0^{(\kappa)}$ is a one-dimensional subspace spanned by 1 and the subspace $A_1^{(\kappa)}$ is spanned by the generators $p_1, \cdots, p_\kappa$. We have

$$A^{(\kappa)} = \sum_{s=0}^{\kappa} A_s^{(\kappa)}.$$

For every

$$\alpha = \Sigma \alpha_s \quad (\alpha_s \in A_s^{(\kappa)})$$

we put

$$\bar{\alpha} = \Sigma (-1)^{\frac{s(s+1)}{2}} \alpha_s.$$

The mapping $\alpha \to \bar{\alpha}$ is an involutive anti-automorphism of the algebra $A^{(\kappa)}$. Using it we define an involutive anti-automorphism $a \to a^*$ of the algebra $\bar{\mathfrak{A}}_2^\nu$ as transposition with conjugation.

We consider the projection operator

$$P: \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \to \begin{pmatrix} \alpha_0 & \beta_0 + \beta_1 \\ \gamma_0 + \gamma_1 & \delta_0 \end{pmatrix}$$

in the algebra $\bar{\mathfrak{A}}_2^\nu$. The subspace $P\bar{\mathfrak{A}}_2^\nu$ may be considered as a matrix algebra with involution if in it we define multiplication by the formula

$$ab = P(a \cdot b).$$

This matrix algebra is a T-algebra of the type $\mathfrak{A}_2^\nu$. The corresponding cone consists of the matrices

$$x = \begin{bmatrix} \rho & \tau \\ \bar{\tau} & \sigma \end{bmatrix}, \tag{8}$$

in which

$$\rho, \sigma \in A_0^{(\kappa)}, \quad \tau \in A_0^{(\kappa)} + A_1^{(\kappa)}$$

and

$$\rho, \sigma > 0, \quad \rho\sigma - \tau\bar{\tau} > 0. \tag{9}$$

This is a $(\nu + 2)$-dimensional spherical cone.

Let $\mathfrak{D}$ denote the subspace in the algebra $\bar{\mathfrak{A}}_2^\nu$ formed by the matrices

$$a = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix},$$

in which

$$\beta, \gamma, \alpha - \delta \in A_0^{(\kappa)} + A_1^{(\kappa)}, \ \alpha + \delta \in A_0^{(\kappa)} + A_2^{(\kappa)}. \tag{10}$$

The dimension of this subspace is easily calculated to be $(\nu+1)(\nu+2)/2+1$. For later use we note that this is also the dimension of the group of automorphisms of the cone V which is the direct product of the one-dimensional similitude group and the group of pseudo-orthogonal transformations leaving invariant the quadratic form (cf. (8))

$$(x, x) = \rho\sigma - \tau\bar{\tau}. \tag{11}$$

We make every matrix $a \in \mathfrak{D}$ correspond to the transformation

$$D_a\colon\ x \longrightarrow a \cdot x + x \cdot a^*$$

of the space of Hermitian matrices of the algebra $\mathfrak{A}_2^\nu$. We shall show in a moment that the space of Hermitian matrices lying in $\mathfrak{A}_2^\nu$ is invariant under all transformations D_a, $a \in \mathfrak{D}$, and that the algebra of derivations of the cone V is the set of restrictions of the transformations D_a to this subspace.

Let

$$x = \begin{pmatrix} \rho & \tau \\ \bar{\tau} & \sigma \end{pmatrix}$$

be a Hermitian matrix in $\mathfrak{A}_2^\nu$ and

$$a = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

a matrix in $\mathfrak{D}$. We have that (cf. (9) and (10))

$$D_a x = \begin{pmatrix} \alpha\rho + \beta\bar{\tau} + \rho\bar{\alpha} + \tau\bar{\beta} & \alpha\tau + \beta\sigma + \rho\bar{\gamma} + \tau\bar{\delta} \\ \gamma\rho + \delta\bar{\tau} + \bar{\tau}\bar{\alpha} + \sigma\bar{\beta} & \gamma\tau + \delta\sigma + \bar{\tau}\bar{\gamma} + \sigma\bar{\delta} \end{pmatrix}$$

$$= \begin{pmatrix} 2\rho\alpha_0 + \beta\bar{\tau} + \tau\bar{\beta} & \sigma\beta + \rho\bar{\gamma} + (\alpha_0 + \delta_0)\tau + \alpha_1\tau + \tau\alpha_1 + \alpha_2\tau - \tau\alpha_2 \\ \dots & 2\sigma\delta_0 + \gamma\tau + \bar{\tau}\bar{\gamma} \end{pmatrix}.$$

The element $\eta = \beta\bar{\tau} + \tau\bar{\beta}$ always lies in $A_0^{(\kappa)} + A_1^{(\kappa)} + A_2^{(\kappa)}$; since $\bar{\eta} = \eta$ this implies that $\eta \in A_0^{(\kappa)}$. In the same way we show that $\gamma\tau + \bar{\tau}\bar{\gamma} \in A_0^{(\kappa)}$. Further

$$\alpha_1\tau + \tau\alpha_1 = 2\tau_0\alpha_1 + \alpha_1\tau_1 + \tau_1\alpha_1 \in A_0^{(\kappa)} + A_1^{(\kappa)}.$$

Finally

$$\xi = \alpha_2\tau - \tau\alpha_2 = \alpha_2\tau_1 - \tau_1\alpha_2 = \alpha_2\delta_1 - \bar{\tau}_1\bar{\alpha}_2 = -\bar{\xi},$$

and, since $\xi \in A_3^{(\kappa)} + A_1^{(\kappa)}$, $\xi \in A_1^{(\kappa)}$. Thus $D_a x \in \mathfrak{A}_2^\nu$.

We now show that the transformation $D_a - (\alpha_0 + \delta_0)$ is a derivation of the quadratic form (11), i.e. that

$$(D_a x, x) = (\alpha_0 + \delta_0)\,(x, x) \tag{12}$$

We have that

$$2(D_a x, x) = (\alpha\rho + \beta\bar{\tau} + \rho\bar{\alpha} + \tau\bar{\beta})\,\sigma + \rho\,(\gamma\tau + \delta\sigma + \bar{\tau}\bar{\gamma} + \sigma\bar{\delta})$$
$$- (\alpha\tau + \beta\sigma + \rho\bar{\gamma} + \tau\bar{\delta})\,\bar{\tau} - \tau\,(\gamma\rho + \delta\bar{\tau} + \bar{\tau}\bar{\alpha} + \sigma\bar{\beta})$$
$$= (\alpha + \bar{\alpha} + \delta + \bar{\delta})\,(\rho\sigma - \bar{\tau}\tau) = 2\,(\alpha_0 + \delta_0)\,(x, x),$$

which proves the result.

It follows from (12) that the transformations D_a, $a \in \mathfrak{D}$, are derivations of the cone V. Dimensional arguments show that every derivation of the cone V is obtained in this way.

Let us sum up the results we have obtained.

Let V be an indecomposable homogeneous selfadjoint cone which is not the exceptional one. Then there exists an associative matrix algebra $\bar{\mathfrak{A}}$ and a projection operator P in the algebra $\bar{\mathfrak{A}}$ such that

1) $P\bar{\mathfrak{A}}_{ij} \subset \bar{\mathfrak{A}}_{ij}$;

2) $Pa^* = (Pa)^*$ *for any* $a \in \bar{\mathfrak{A}}$.

3) *the subspace $\mathfrak{A} = P\bar{\mathfrak{A}}$ of the algebra $\bar{\mathfrak{A}}$, equipped with the induced gradation and involution and with a multiplication defined by the formula*

$$ab = P(a \cdot b) \quad (a,\ b \in \mathfrak{A}),$$

is a T-algebra corresponding to the cone V.

Further, there exists a subspace $\mathfrak{D} \subset \bar{\mathfrak{A}}$ such that, for every Hermitian matrix $x \in \mathfrak{A}$, $D_a x = a \cdot x + x \cdot a^* \in \mathfrak{A}$ for every $a \in \mathfrak{D}$ and the mapping $a \to D_a$ is an isomorphic mapping of the space $\mathfrak{D}$ onto the algebra of derivations of the cone V.

(For the cones of types I *and* III, *$\mathfrak{D}$ must be taken to be the algebra $\bar{\mathfrak{A}}$ itself; for the cone of type* II *it must be taken to be the set of complex matrices with real trace.)*

We observe that

$$[D_a, D_b] = D_{a\cdot b - b\cdot a}.$$

CHAPTER III

Modification of the Matrix Calculus

§1. The decomposition of exceptional algebras

Let $\mathfrak{A}$ be a T-algebra of rank m and $\mathfrak{A}^c$ it kernel. The algebra $\mathfrak{A}^c$ splits into the direct sum of T-algebras

$$\mathfrak{A}_\alpha = \sum_{i,j \in M_\alpha} \mathfrak{A}_{ij} \qquad (\alpha = 1, \ldots, \mu)$$

(cf. the definitions in §4 of Chapter I). Each of the T-algebras $\mathfrak{A}_\alpha$ satisfies the conditions of Proposition 3 and therefore is isomorphic to one of the T-algebras

$$\mathfrak{A}^1_m,\ \mathfrak{A}^2_m,\ \mathfrak{A}^4_m,\ \mathfrak{A}^8_3,\ \mathfrak{A}^\nu_2 \qquad (\nu \geqslant 3).$$

The rank of the T-algebra $\mathfrak{A}_\alpha$ is m_α while for $i,\ j \in M_\alpha,\ i \neq j$, the dimension of the space $\mathfrak{A}_{ij}$ is $\nu_{\alpha\alpha}$.

With every T-algebra $\mathfrak{A}_\alpha$ is connected the algebra

$$S_{\alpha\alpha} = \begin{cases} A^{\nu_{\alpha\alpha}}, & \text{if } m_\alpha \neq 2 \text{ or } \nu_{\alpha\alpha} \leqslant 2, \\ A^{(\varkappa)}, & \text{if } m_\alpha = 2 \text{ and } \nu_{\alpha\alpha} = \varkappa + 1 \geqslant 3, \end{cases}$$

and the mappings

$$\theta^{ij}\colon\ \mathfrak{A}_{ij} \to S_{\alpha\alpha} \quad (i,\ j \in M_\alpha)$$

in such a way that if, for every $\lambda \in S_{\alpha\alpha}$, we put

$$\operatorname{Pr}\lambda = \begin{cases} \lambda & \text{when } S_{\alpha\alpha} = A^{\nu_{\alpha\alpha}}, \\ \lambda_0 + \lambda_1 & \text{when } S_{\alpha\alpha} = A^{(\varkappa)} \end{cases} \tag{1}$$

(cf. the definitions in §3, Chapter II) then, for $i,\ j,\ k \in M_\alpha$,

$$\theta^{ik}(a_{ij}b_{jk}) = \begin{cases} \operatorname{Pr}(\theta^{ij}(a_{ij})\,\theta^{jk}(b_{jk})), & i \neq k, \\ \operatorname{Re}(\theta^{ij}(a_{ij})\,\theta^{jk}(b_{jk})), & i = k, \end{cases} \tag{2}$$

$$\theta^{ji}(a^*_{ij}) = \overline{\theta^{ij}(a_{ij})} \tag{3}$$

(cf. Chapter II). We denote the inverse image of the unit element of the algebra $S_{\alpha\alpha}$ under the mapping θ^{ij} by e_{ij}.

For every $\beta \neq \alpha$ we now define the mapping

$$\theta^{ij}_{kl}\colon \mathfrak{A}_{ij} \to \mathfrak{A}_{kl}, \quad i,\ k \in M_\alpha; \qquad j,\ l \in M_\beta,$$

by the formula

$$\theta^{ij}_{kl}(a_{ij}) = e_{ki}a_{ij}e_{jl}.$$

(We have not inserted brackets since, in this case, by Proposition 2, we have associativity.) It is not difficult to show that

$$\theta^{kl}_{pq}\,\theta^{ij}_{kl} = \theta^{ij}_{pq}.$$

This implies that we may take a standard linear space $S_{\alpha\beta}$ of dimension $\nu_{\alpha\beta}$ and construct the isomorphic mappings

$$\theta^{ij}\colon\ \mathfrak{A}_{ij} \to S_{\alpha\beta}, \quad i \in M_\alpha; \quad j \in M_\beta,$$

so that

$$\theta^{kl}\theta^{ij}_{kl} = \theta^{ij}$$

for any $i,\ k \in M_\alpha,\ j,\ l \in M_\beta$.

We assume that $m_\alpha \neq 2$ or $\nu_{\alpha\alpha} \leq 2$ and introduce into the space $S_{\alpha\beta}$ the structure of a left $S_{\alpha\alpha}$-module so that for any $i,\ j \in M_\alpha,\ k \in M_\beta$,

$$\theta^{ik}(a_{ij}b_{jk}) = \theta^{ij}(a_{ij})\,\theta^{jk}(b_{jk}).$$

Under the above assumptions $\theta^{ij}(\mathfrak{A}_{ij}) = S_{\alpha\alpha}$ $(i,\ j \in M_\alpha,\ i \neq j)$ and equation (1) may be adopted as a definition. However, we must show that it is not inconsistent. We only consider the following, most difficult case:

$$\theta^{ij}(a_{ij}) = \theta^{ji}(g_{ji}),\quad \theta^{jk}(b_{jk}) = \theta^{ik}(h_{ik}),$$

where we must show that

$$\theta^{ik}(a_{ij}b_{jk}) = \theta^{jk}(g_{ji}h_{ik}).$$

We first observe that

$$g_{ji} = 2(e_{ji}a_{ij})\,e_{ji} - a^*_{ij},$$

since

$$\theta^{ji}(2(e_{ji}a_{ij})\,e_{ji} - a^*_{ij}) = 2(\varepsilon,\ \theta^{ij}(a_{ij}))\varepsilon - \overline{\theta^{ij}(a_{ij})} = \theta^{ij}(a_{ij}).$$

(cf. Chapter II, (4)). We now have that

$$g_{ji}h_{ik} = 2((e_{ji}a_{ij})\,e_{ji})(e_{ij}b_{jk}) - a^*_{ij}(e_{ij}b_{jk})$$
$$= 2(e_{ji}a_{ij})\,b_{jk} - a^*_{ji}(e_{ij}b_{jk}) = e_{ji}(a_{ij}b_{jk}),$$

which proves the result. There is no difficulty in verifying the axioms for a module.

If $\nu_{\alpha\alpha} = 8$, i.e. $S_{\alpha\alpha}$ is a Cayley algebra, then the space $S_{\alpha\beta}$ must be zero-dimensional for all $\beta \neq \alpha$, since the Cayley algebra is nonassociative and does not admit nontrivial linear representations.

We shall say that a T-algebra $\mathfrak{A}$ and the corresponding cone are *classical* if the T-algebras $\mathfrak{A}_\alpha$ contain no T-algebras of the type $\mathfrak{A}_3^8$.

From the above it is clear that we have

Theorem 1. *Every homogeneous convex cone splits into the direct sum of a classical homogeneous convex cone and some exceptional 27-dimensional cones.*

§2. The "skeleton" of a T-algebra

Let $\mathfrak{A}$ be a classical T-algebra. We consider the direct sum

$$S = \sum_{\alpha,\beta} S_{\alpha\beta}$$

and continue the constructions of the previous section so as to make S a matrix algebra with involution in such a way that the following relations hold:

$$\theta^{ik}(a_{ij}b_{jk}) = \begin{cases} \theta^{ij}(a_{ij})\,\theta^{jk}(b_{jk}), & \text{if } \alpha(i) \neq \alpha(k), \\ \Pr(\theta^{ij}(a_{ij})\,\theta^{jk}(b_{jk})), & \text{if } \alpha(i) = \alpha(k), \text{ but } i \neq k, \\ \operatorname{Re}(\theta^{ij}(a_{ij})\,\theta^{jk}(b_{jk})), & \text{if } i = k, \end{cases}$$

$$\theta^{ji}(a_{ij}^{*}) = \theta^{ij}(a_{ij})^{*}$$

and so that the subalgebra $\Sigma S_{\alpha\alpha}$ should belong to the "associative center" of the algebra S, i.e. so that $u(vw) = (uv)w$ if at least one of the elements u, v, w lies in $\Sigma S_{\alpha\alpha}$.

We have already done this in part in Chapter II, where we defined an operation of multiplication in the spaces $S_{\alpha\alpha}$, and in §1 of this chapter, where the space $S_{\alpha\beta}$ was provided with the structure of a left $S_{\alpha\alpha}$-module for $m_\alpha \neq 2$ or $\nu_{\alpha\alpha} \leq 2$ and $\beta \neq \alpha$.

Suppose now that $m_\alpha = 2$, $\nu_{\alpha\alpha} = \kappa + 1$, $\kappa \geq 2$. As in Chapter II let $p_1, \cdots, p_\kappa$ be the generators of the algebra $S_{\alpha\alpha} = A^{(\kappa)}$. Multiplication by $p_1, \cdots, p_\kappa$ in the space $S_{\alpha\beta}$ is defined by the relation (2). (That it is noninconsistent is proved in the same way as in §1.) To define the structure of an $A^{(\kappa)}$-module in the space $S_{\alpha\beta}$ we must have the following for $u \in S_{\alpha\beta}$:

$$p_i(p_i u) = -u,$$
$$p_i(p_j u) = -p_j(p_i u) \quad (i \neq j).$$

Let us verify the first of these identities, for example. Let $M_\alpha = \{r, s\}$, $s \in M_\beta$ and

$$\theta^{rs}(a_{rs}) = p_i, \quad \theta^{st}(b_{st}) = u.$$

Then $\theta^{sr}(a_{rs}^{*}) = -p_i$ and

$$p_i(p_i u) = -\theta^{st}(a_{rs}^{*}(a_{rs}b_{st})) = -\theta^{st}((a_{rs}^{*}a_{rs})\,b_{st}) = -\theta^{st}(b_{st}) = -u.$$

Thus in all cases the space $S_{\alpha\beta}$ has the structure of a left $S_{\alpha\alpha}$-module. In the same way it can be proved with the structure of a right $S_{\beta\beta}$-module and we easily verify the identity

$$\lambda(u\mu) = (\lambda u)\mu$$

for all $\lambda \in S_{\alpha\alpha}$, $\mu \in S_{\beta\beta}$ and $u \in S_{\alpha\beta}$.

We must still define the multiplication of elements in $S_{\alpha\beta}$ $(\alpha \neq \beta)$ by elements in $S_{\beta\gamma}$ $(\beta \neq \gamma)$, the result lying in $S_{\alpha\gamma}$, in such a way that the relation (2) and the identities

$$\lambda(uv) = (\lambda u)v, \tag{4}$$

$$u(\mu v) = (u\mu)v, \tag{5}$$

$$u(v\nu) = (uv)\nu \tag{6}$$

hold for $u \in S_{\alpha\beta}$, $v \in S_{\beta\gamma}$, $\lambda \in S_{\alpha\alpha}$, $\mu \in S_{\beta\beta}$, $\nu \in S_{\gamma\gamma}$. All this is done very simply, except in the case when $\alpha = \gamma$ and $S_{\alpha\alpha}$ is a Clifford algebra. We now consider this case.

Thus, let $S_{\alpha\alpha} = A^{(\kappa)}$, $\kappa \geq 2$. The relation (2) is not enough for us to be able to define an operation of multiplication of elements in the space $S_{\alpha\beta}$ by elements in the space $S_{\beta\alpha}$. For $u \in S_{\alpha\beta}$, $v \in S_{\beta\alpha}$ the relation (2) only defines the projection of the element $uv \in A^{(\kappa)}$ onto $A_0^{(\kappa)} + A_1^{(\kappa)}$; in particular it defines $\operatorname{Re} uv$. If ρ is an arbitrary element of the algebra $A^{(\kappa)}$, then (4) implies that the following equation must hold:

$$\operatorname{Re}{}_{\rho}(uv) = \operatorname{Re}(\rho u)\, v.$$

Since the right-hand side of this equation is known from (2) it may be taken as a definition of the element uv.

We now show that this definition agrees with (2) for $i \neq k$. Let

$$u = \theta^{ij}(a_{ij}),\ v = \theta^{jk}(b_{jk}) \quad (i, k \in M_\alpha;\ j \in M_\beta).$$

To prove the equation

$$\theta^{ik}(a_{ij}b_{jk}) = \operatorname{Pr}(uv)$$

it is sufficient to verify that for $s = 1, \cdots, \kappa$

$$\operatorname{Re} p_s \theta^{ik}(a_{ij}b_{jk}) = \operatorname{Re}(p_s u)\, v.$$

Let $p_s = \theta^{ki}(c_{ki})$. Then

$$\operatorname{Re} p_s\theta^{ik}(a_{ij}b_{jk}) = \theta^{kk}(c_{ki}(a_{ij}b_{jk})) = \theta^{kk}((c_{ki}a_{ij})\, b_{jk})$$
$$= \operatorname{Re}\theta^{kj}(c_{ki}a_{ij})\,\theta^{jk}(b_{jk}) = \operatorname{Re}(p_s u)\, v$$

We now verify that the identity (4) holds. For any $\rho \in A^{(\kappa)}$:

$$\operatorname{Re}\rho(\lambda(uv)) = \operatorname{Re}(\rho\lambda)(uv) = \operatorname{Re}((\rho\lambda)u)\, v = \operatorname{Re}(\rho(\lambda u))\, v = \operatorname{Re}\rho((\lambda u)v),$$

which implies that $\lambda(uv) = (\lambda u)v$.

To establish the identity (5) we first observe that (2) and the associativity relations in the algebra $\mathfrak{A}$ imply that

$$\operatorname{Re} u(\mu v) = \operatorname{Re}(u\mu)\, v$$

for all $u \in S_{\alpha\beta}$, $v \in S_{\beta\alpha}$, $\mu \in S_{\beta\beta}$. Therefore, for any $\rho \in A^{(\kappa)}$

$$\operatorname{Re}\rho(u(\mu v)) = \operatorname{Re}(\rho u)(\mu v) = \operatorname{Re}((\rho u)\mu)\, v = \operatorname{Re}(\rho(u\mu))\, v = \operatorname{Re}\rho((u\mu)v),$$

so that $u(\mu v) = (u\mu)v$. The identity (6) may be checked in the same way.

We define an involution in the algebra S as follows: for $\lambda \in S_{\alpha\alpha}$ we put $\lambda^* = \bar{\lambda}$, and for $u \in S_{\alpha\beta}$, $\alpha \neq \beta$, we define u^* by (3). It is easy to check that this definition is not inconsistent and has the properties of an involution.

Clearly the algebra S and the numbers m_α uniquely determine the T-algebra

$\mathfrak{A}$. We shall call S the *skeleton* of the T-algebra $\mathfrak{A}$.

The results we have obtained, together with the identity (1) of Chapter I, allow us to prove the following interesting theorem which, however, we shall not use later.

Theorem 2. *If $m_\alpha = 2$ and $\nu_{\alpha\alpha} > 4$ then $S_{\alpha\beta} = 0$ for all $\beta < \alpha$ or for all $\beta > \alpha$. The same is true for $\nu_{\alpha\alpha} = 4$ if at least one of the $S_{\alpha\alpha}$-modules $S_{\alpha\beta}$ $(\beta \neq \alpha)$ is exact.*

We observe that the factor algebra of the algebra $A^{(3)}$ by any of its nontrivial ideals is isomorphic to the algebra of quaternions.

§3. The completion of a T-algebra

As before we shall assume that the T- algebra $\mathfrak{A}$ is classical. We imbed each of the subspaces $\mathfrak{A}_{ij} \subset \mathfrak{A}$ in the linear space $\overline{\mathfrak{A}}_{ij}$ of the same dimension as $S_{\alpha(i)\,\alpha(j)}$ and extend the mapping θ^{ij} to an isomorphic mapping of the space $\overline{\mathfrak{A}}_{ij}$ onto $S_{\alpha(i)\,\alpha(j)}$. (Clearly, $\overline{\mathfrak{A}}_{ij}$ fails to coincide with $\mathfrak{A}_{ij}$ only when $\alpha(i) = \alpha(j)$.) We next form the direct sum of the linear spaces

$$\overline{\mathfrak{A}} = \sum \overline{\mathfrak{A}}_{ij}.$$

For any elements $a_{ij} \in \overline{\mathfrak{A}}_{ij}$, $b_{jk} \in \overline{\mathfrak{A}}_{jk}$ we define their product $a_{ij} \cdot b_{jk} \in \overline{\mathfrak{A}}_{ik}$ so that

$$\theta^{ik}(a_{ij} \cdot b_{jk}) = \theta^{ij}(a_{ij})\, \theta^{jk}(b_{jk}).$$

For $a, b \in \overline{\mathfrak{A}}$ we put

$$a \cdot b = \sum a_{ij} \cdot b_{jk},$$

where a_{pq}, b_{pq} are the projections of the elements a, b onto $\overline{\mathfrak{A}}_{pq}$, and we thus convert the space $\overline{\mathfrak{A}}$ into a matrix algebra. We provide this algebra with an involution* by the formulae

$$\theta^{ji}(a_{ij}^*) = \overline{\theta^{ij}(a_{ij})},$$

$$a^* = \sum a_{ij}^* \quad (a = \sum a_{ij} \in \overline{\mathfrak{A}}).$$

The algebra $\mathfrak{A}$ is included in $\overline{\mathfrak{A}}$ as a subspace. Also

$$\mathfrak{A}_{ij} \subset \overline{\mathfrak{A}}_{ij};$$

the involution in $\mathfrak{A}$ is the restriction of the involution in $\overline{\mathfrak{A}}$; as regards multiplication, it is clear from (2) that there exists a projection P of the space $\overline{\mathfrak{A}}$ onto the subspace $\mathfrak{A}$ so that

$$a \cdot b = P(a \cdot b)$$

for any $a, b \in \mathfrak{A}$. The projection P is defined as follows:

$$\theta^{ij}(Pa_{ij}) = \begin{cases} \theta^{ij}(a_{ij}), & \text{if } \alpha(i) \neq \alpha(j), \\ \Pr\theta^{ij}(a_{ij}), & \text{if } \alpha(i) = \alpha(j), \text{ but } i \neq j, \\ \operatorname{Re}\theta^{ij}(a_{ij}), & \text{if } i = j. \end{cases}$$

Clearly

$$P\overline{\mathfrak{A}}_{ij} = \mathfrak{A}_{ij};$$
$$Pa^* = (Pa)^* \quad (a \in \overline{\mathfrak{A}}).$$

The matrix algebra with involution $\overline{\mathfrak{A}}$ will be called the *completion* of the T-algebra $\mathfrak{A}$.

The subalgebra

$$\overline{\mathfrak{A}}_\alpha = \sum_{i,j \in M_\alpha} \overline{\mathfrak{A}}_{ij}$$

is isomorphic to the associative matrix algebra $\overline{\mathfrak{A}}_{m_\alpha}^{\nu_{\alpha\alpha}}$ (cf. Chapter II). The T-algebra $\mathfrak{A}_\alpha$ is identified with the T-algebra $\mathfrak{A}_{m_\alpha}^{\nu_{\alpha\alpha}}$ under this isomorphism.

We put

$$\overline{\mathfrak{A}}^c = \Sigma\, \overline{\mathfrak{A}}_\alpha;$$

then

$$\overline{\mathfrak{A}} = \overline{\mathfrak{A}}^c + \mathfrak{A}^u,$$

where

$$\mathfrak{A}^u = \sum_{i \not\equiv j \,(\mathrm{mod}\, \overline{R})} \mathfrak{A}_{ij}.$$

It follows from the results of §2 that the subalgebra $\overline{\mathfrak{A}}^c$ belongs to the associative center of the algebra $\overline{\mathfrak{A}}$. We have the following inclusions

$$\overline{\mathfrak{A}}^c \cdot \mathfrak{A}^u \subset \mathfrak{A}^u, \quad \mathfrak{A}^u \cdot \overline{\mathfrak{A}}^c \subset \mathfrak{A}^u.$$

Further, if $a \in \mathfrak{A}^c$, $p \in \mathfrak{A}^u$, then

$$a \cdot p = ap, \quad p \cdot a = pa. \tag{7}$$

We provide the algebra $\overline{\mathfrak{A}}$ with a Euclidean metric by means of the formula

$$(a, b) = \operatorname{Sp} P(a \cdot b^*).$$

On the subspace $\mathfrak{A}$ it coincides with the metric introduced in §3 of Chapter I. It is also easy to check that

$$(a^*, b^*) = (a, b) \tag{8}$$

and that the projection P is symmetric in this metric.

CHAPTER IV

The structure of the full group of automorphisms

§1. Extension of the group of automorphisms

Proposition 4. *Let V be a homogeneous convex cone, G a connected transitive group of its automorphisms and x_0 an arbitrary point of V. Let G_1 be a connected linear group containing the group G. Let K_1 denote the set of infinitesimal linear transformations belonging to the Lie algebra of the group G_1 and becoming 0 at the point x_0. If all the transformations in K_1 have zero trace then all the transformations in the group G_1 are automorphisms of the cone V.*

Proof. The orbit V_1 of the group G_1 passing through the point x_0 is a connected open cone containing the cone V. We shall show that $V_1 = V$.

Let $\mathfrak{K}_1$ be the subgroup of the group $\mathfrak{G}_1$ formed by those transformations which leave the point x_0 fixed and let $\mathfrak{K}_0$ be the connected component of the unit element of the group $\mathfrak{K}_1$. Clearly the Lie algebra of the group K_0 coincides with K_1. Therefore the transformations in $\mathfrak{K}_0$ are unimodular and there exists an invariant measure in the homogeneous space $\mathfrak{G}_1/\mathfrak{K}_0$.

If $V_1 \neq V$, there exists a connected simply connected open cone $V_2 \subset V_1$, containing the cone V and also at least one point of its boundary. Let π denote the natural covering mapping of the space $\widetilde{V}_1 = \mathfrak{G}_1/\mathfrak{K}_0$ onto the cone V_1. The covering π is trivial over V_2. Let $\widetilde{V}_2$ be one of the connected components of the set $\pi^{-1}(V_2)$. The restriction of the mapping π to $\widetilde{V}_2$ is a homeomorphism of $\widetilde{V}_2$ onto V_2. The invariant measure on $\mathfrak{G}_1/\mathfrak{K}_0$ induces some measure on $\widetilde{V}_2$. We transfer this measure onto V_2 by means of the mapping π and let ψ denote the density of the measure obtained on $\widetilde{V}_2$ (relative to the measure which is invariant under parallel translations).

Now let $\widetilde{x}_0$ be the inverse image of the point $x_0 \in V_2$ under the mapping π, lying in $\widetilde{V}_2$. The set $\mathfrak{G}\widetilde{x}_0$ lies in $\pi^{-1}(V_2)$ and is connected. Therefore, $\mathfrak{G}\widetilde{x}_0 \subset \widetilde{V}_2$. The definition of the function ψ implies that if $C \in \widetilde{\mathfrak{G}}$ then

$$\psi(Cx_0) = \frac{\psi(x_0)}{\det C}.$$

This equation shows that at the points of the cone V the function ψ coincides with the characteristic function of the cone V. It is well known (cf. §2 of Chapter I) that the characteristic function of a cone V grows without bound on approach to its boundary; however, the function ψ is defined and continuous in the cone V_2 containing a boundary point of the cone V. This contradiction shows that $V_1 = V$.

§2. The quasitriangular group of automorphisms

Let V be a homogeneous convex cone, $\mathfrak{A}$ the corresponding T-algebra, $\mathfrak{A}^c = \Sigma\mathfrak{A}_\alpha$ its kernel (cf. the definitions in §1 of Chapter III). The convex cone

$$V^c = V \cap \mathfrak{A}^c$$

will be called the *kernel of the cone* V.

It is easy to see that if $t \in \mathfrak{A}$ is a triangular matrix with positive elements on the main diagonal and $tt^* \in \mathfrak{A}^c$ then $t \in \mathfrak{A}^c$. This implies that V^c is a homogeneous selfadjoint cone corresponding to the T-algebra $\mathfrak{A}^c$. It splits into the direct sum of cones V_α, corresponding to the T-algebras $\mathfrak{A}_\alpha$ $(\alpha = 1, \cdots, \mu)$.

We now assume that the cone V is classical and let $\overline{\mathfrak{A}}$ be the completion of the T-algebra $\mathfrak{A}$ (cf. §3 of Chapter III). For every $\alpha \in M$ we construct the subspace $\mathfrak{D}_\alpha \subset \overline{\mathfrak{A}}_\alpha$ according to §3 of Chapter II and we form the direct sum

$$\mathfrak{D} = \Sigma\mathfrak{D}_\alpha \subset \overline{\mathfrak{A}}^c.$$

For $a \in \mathfrak{D}$ we put

$$D_a x = a \cdot x + x \cdot a^* \quad (x \in \mathfrak{X}). \tag{1}$$

Clearly

$$[D_a, D_b] = D_{a\cdot b - b\cdot a} \quad (a, b \in \mathfrak{D}). \tag{2}$$

We denote the Lie algebra formed by the transformations D_a, $a \in \mathfrak{D}$, by G^c. The restriction of the algebra G^c to the invariant subspace $\mathfrak{X}^c = \mathfrak{X} \cap \mathfrak{A}^c$ coincides with the algebra of derivations of the cone V^c.

We denote the space of Hermitian matrices of the algebra $\overline{\mathfrak{A}}$ by $\overline{\mathfrak{X}}$.

Lemma 1. *For every* $a \in \mathfrak{D}$ *the transformation*

$$\overline{D}_a:\ x \longrightarrow a \cdot x + x \cdot a^* \quad (x \in \overline{\mathfrak{X}})$$

of the space $\mathfrak{X}$ *commutes with the projection* P *(cf. §3 of Chapter* III).

Proof. For $a \in \mathfrak{D}$, $x, y \in \overline{\mathfrak{X}}$ we have, by virtue of the formula (8) of Chapter III, that

$$(a\cdot x, y) = (x^*\cdot a^*, y^*) = \operatorname{Sp} P(x^*\cdot a^*\cdot y) = (x^*, y^*\cdot a) = (x, a^*\cdot y),$$
$$(x\cdot a^*, y) = \operatorname{Sp} P(x\cdot a^*\cdot y^*) = (x, y\cdot a),$$

hence

$$(\overline{D}_a x, y) = (x, \overline{D}_{a^*} y), \tag{3}$$

i.e. the transformation $\overline{D}_{a^*}$ is adjoint to $\overline{D}_a$. Since the transformations $\overline{D}_a$ and $\overline{D}_{a^*}$ leave the subspace $\mathfrak{X} = P\overline{\mathfrak{X}}$ invariant, this implies that they also leave the orthogonal complement invariant and, thus, commute with p.

We now put

$$\mathfrak{T}^u = \mathfrak{T} \cap \mathfrak{A}^u.$$

(We recall that $\mathfrak{T}$ is the space of triangular matrices of the algebra $\mathfrak{A}$.) The transformations

$$D_t: \ x \longrightarrow tx + xt^*, \quad t \in \mathfrak{T}^u,$$

of the space $\mathfrak{X}$ form a Lie algebra which we shall denote by T^u.

We consider the sum

$$G^q = G^c + T^u$$

and show that it is closed under commutation. This follows from the formula

$$[D_a, D_t] = D_{a\cdot t - t\cdot a} \quad (a \in \mathfrak{D}, \ t \in \mathfrak{T}^u). \tag{4}$$

(It is easy to see that the element $a \cdot t - t \cdot a$ belongs to $\mathfrak{T}^u$.)

For any $x \in \mathfrak{X}$, Lemma 1 implies that

$$[D_a, D_t]\, x = D_a P(t\cdot x + x\cdot t^*) - P(t\cdot D_a x + D_a x\cdot t^*)$$
$$= P(\overline{D}_a(t\cdot x + x\cdot t^*) - t\cdot D_a x - D_a x\cdot t^*) = P((a\cdot t - t\cdot a)\cdot x$$
$$+ x\cdot(t^*\cdot a^* - a^*\cdot t^*)) = D_{a\cdot t - t\cdot a}x,$$

which is what we had to show.

It is easy to verify that when $t \in \mathfrak{T} \cap \mathfrak{A}^c$ the transformation D_t of the space $\mathfrak{X}$, understood in the sense of §1 of Chapter I, coincides with the transformation D_t defined by the formula (1) for t an element of the space $\mathfrak{D}$. Therefore, the Lie algebra G^q contains the Lie algebra

$$T = \{D_t: \ t \in \mathfrak{T}\},$$

generating the simply-transitive group of automorphisms of the cone V.

Proposition 5. *The connected linear group G^q, generated by the linear Lie algebra G^q, leaves the cone V invariant and acts transitively on it.*

Proof. By virtue of the remark made above, the group G^q, contains the transitive triangular group of automorphisms of the cone V. Proposition 5 will follow from Proposition 4 if we show that every transformation $A \in G^q$ for which $Ae = 0$ has zero trace.

Let $a \in \mathfrak{A}^c$, $t \in \mathfrak{T}^u$ and

$$(D_a + D_t)\, e = (a + a^*) + (t + t^*) = 0.$$

Since $a + a^* \in \mathfrak{A}^c$, $t + t^* \in \mathfrak{A}^u$, this implies that $t = 0$ or $a + a^* = 0$. By virtue of (3) $D_a^* = -D_a$, i.e. the transformation D_a is skew-symmetric. We have that

$$\mathrm{Sp}\,(D_a + D_t) = \mathrm{Sp}\, D_a = 0,$$

which is what we had to prove.

The group G^q may be called the *quasi-triangular group of automorphisms of* *the cone* V. It clearly contains the connected Lie group $\mathfrak{G}^c$ generated by the Lie algebra G^c. The mapping which puts every transformation $C \in G^c$ in correspondence with its restriction to $\mathfrak{X}^c$ is a homomorphism of the group G^c onto the connected component of the unit element of the group of automorphisms of the cone V^c. The kernel of this homomorphism is finite.

§3. Invariance of the kernel

In this section we shall prove

Theorem 3. *The kernel of a homogeneous convex cone* V, *corresponding to the T-algebra* $\mathfrak{A}$, *is invariant under all the automorphisms of the cone* V *which leave the point* e *(the unit matrix) invariant.*

We preface the proof of the theorem by a number of lemmas. We shall assume throughout that the cone V is classical but, as is easily seen, Theorem 3 holds for any cone.

For $x, y \in \mathfrak{X}$ we put

$$R_x y = x \square y = \frac{1}{2}(xy + yx).$$

It is easy to check that if $x \in \mathfrak{X}^c$ then

$$R_x y = \frac{1}{2}(x \cdot y + y \cdot x)$$

and thus $R_x \in G^c$ (cf. §2). Let

$$\widetilde{\mathfrak{X}}^c = \{x \in \mathfrak{X} :\ R_x \in G\}$$

where G is the algebra of derivations of the cone V.

The proof of Theorem 3 will be based on the fact that $\widetilde{\mathfrak{X}}^c = \mathfrak{X}^c$.

Let C be an automorphism of the cone V leaving the point $e \in V$ fixed. The transformation C is an automorphism of the connectedness algebra of the cone V at the point e (cf. §2 of Chapter I) and therefore for any $x \in \mathfrak{X}$

$$R_{Cx} = CR_x C^{-1}.$$

This implies that $C\widetilde{\mathfrak{X}}^c \subset \widetilde{\mathfrak{X}}^c$. Thus the subspace $\widetilde{\mathfrak{X}}^c \subset \mathfrak{X}$ is invariant under all automorphisms of the cone V which leave the point e fixed and for our purpose it is in fact sufficient to prove that $\widetilde{\mathfrak{X}}^c = \mathfrak{X}^c$.

For any Hermitian matrix $x = (x_{ij}) \in \mathfrak{A}$ we put

$$\widehat{x} = \frac{1}{2}\sum x_{ii} + \sum_{i<j} x_{ij},$$

$$\underset{\smile}{x} = \frac{1}{2}\sum x_{ii} + \sum_{i>j} x_{ij}.$$

The triangular matrix $\hat{x}$ corresponds to the derivation

$$D_{\hat{x}}\colon\ y \to \hat{x}y + y\underset{\vee}{x}$$

of the cone V. It is easy to see that

$$2(D_{\hat{x}} - R_x)y = (\hat{x} - \underset{\smile}{x})y - y(\hat{x} - \underset{\smile}{x})$$

and that

$$(D_{\hat{x}} - R_x)e = 0.$$

By virtue of the remarks made above this implies

Lemma 2. *The subspace* $\tilde{\mathfrak{X}}^c \subset \mathfrak{X}$ *is invariant under the transformations*

$$K_x\colon\ y \to (\hat{x} - \underset{\smile}{x})y - y(\hat{x} - \underset{\smile}{x}), \qquad x \in \tilde{\mathfrak{X}}^c.$$

We now put

$$\mathfrak{X}_{ij} = \mathfrak{X} \cap (\mathfrak{A}_{ij} + \mathfrak{A}_{ji}).$$

Lemma 3. *The space* $\tilde{\mathfrak{X}}^c$ *equals the sum of its intersections with the spaces* $\mathfrak{X}_{ij}$.

Proof. Let x be an arbitrary element of $\tilde{\mathfrak{X}}^c$. Let e_i denote the diagonal matrix with unity at the ith place and zeros elsewhere. Then $e_i \in \mathfrak{X}^c \subset \tilde{\mathfrak{X}}^c$. By virtue of Lemma 2

$$K_x e_i = \sum_{k<i}(x_{ki} + x_{ik}) - \sum_{k>i}(x_{ki} + x_{ik}) = y \in \tilde{\mathfrak{X}}^c.$$

Further, for $j \neq i$,

$$K_y e_j = -(x_{ij} + x_{ji}) \in \tilde{\mathfrak{X}}^c,$$

which is what we had to prove.

Lemma 4. *If* $\tilde{\mathfrak{X}}^c \cap \mathfrak{X}_{ij} \neq 0$, *then* $n_{ik} = n_{jk}$ *for all* $k \neq i, j$.

Proof. Let $x \in \tilde{\mathfrak{X}}^c \cap \mathfrak{X}_{ij}$, $x \neq 0$. The transformation $K_x = 2(D_{\hat{x}} - R_x)$ is a derivation of the connectedness algebra of the cone V at the point e. Therefore, for any a_{jk}

$$2(a_{jk} + a_{jk}^*) \square K_x(a_{jk} + a_{jk}^*) = K_x(a_{jk}a_{jk}^* + a_{jk}^*a_{jk}).$$

If i, j, k are distinct, then the projection of this equation onto $\mathfrak{A}_{ij}$ yields

$$(x_{ij}a_{jk})a_{jk}^* = x_{ij}(a_{jk}a_{jk}^*) = \rho(a_{jk}a_{jk}^*)x_{ij}.$$

Multiplying both sides on the right by x_{ij}^* we find that

$$((x_{ij}a_{jk})a_{jk}^*)x_{ij}^* = (x_{ij}a_{jk})(x_{ij}a_{jk})^* = \rho(a_{jk}a_{jk}^*)x_{ij}x_{ij}^*,$$

i.e.

$$\rho((x_{ij}a_{jk})(x_{ij}a_{jk})^*) = \rho(a_{jk}a_{jk}^*)\rho(x_{ij}x_{ij}^*). \tag{5}$$

This equation shows that $n_{ik} \geq n_{jk}$. In the same way we show that $n_{jk} \geq n_{ik}$.

Lemma 5. *Let $i<k<j$. If $\widetilde{\mathfrak{X}}^c \cap \mathfrak{X}_{ij} \neq 0$ and $n_{ik} \neq 0$ then $\widetilde{\mathfrak{X}}^c \cap \mathfrak{X}_{ik} \neq 0$ and $\widetilde{\mathfrak{X}}^c \cap \mathfrak{X}_{kj} \neq 0$.*

Proof. Let $x = x_{ij} + x_{ji}$ be a nonzero element in $\widetilde{\mathfrak{X}}^c$. For every $t = t_{kj}$ the transformation

$$D = 2[R_x, D_t] + D_{xt^*}$$

is a derivation of the cone V. Moreover

$$De = x(t+t^*) + (t+t^*)x - 2(tx + xt^*) + tx + xt^* = xt + t^*x = 0.$$

Therefore

$$-De_i = D_t x = t_{kj}x_{ji} + x_{ij}t^*_{kj} \in \widetilde{\mathfrak{X}}^c.$$

If $t \neq 0$, then, as is clear from (5), $x_{ij}t^*_{kj} \neq 0$, so that $\widetilde{\mathfrak{X}}^c \cap \mathfrak{X}_{ik} \neq 0$. In the same way we show that $\widetilde{\mathfrak{X}}^c \cap \mathfrak{X}_{kj} \neq 0$.

In the terminology of §4 of Chapter I Lemmas 4 and 5 may be formulated as follows: if $\widetilde{\mathfrak{X}}^c \cap \mathfrak{X}_{ij} \neq 0$ then $i \equiv j \pmod{\widetilde{R}}$ and for any k lying between i and j either $n_{ik} = n_{kj} = 0$ or $\widetilde{\mathfrak{X}}^c \cap \mathfrak{X}_{ik} \neq 0$ and $\widetilde{\mathfrak{X}}^c \cap \mathfrak{X}_{kj} \neq 0$. From this we easily deduce by induction on $|i-j|$ that if $\widetilde{\mathfrak{X}}^c \cap \mathfrak{X}_{ij} \neq 0$ then $i \equiv j \pmod{\overline{R}}$ (cf. §4 of Chapter I for the inductive definition of the relation $\overline{R}$). This means that $\widetilde{\mathfrak{X}}^c \subset \mathfrak{X}^c$. Using the opposite inclusion, proved at the beginning of this section, we find that $\widetilde{\mathfrak{X}}^c = \mathfrak{X}^c$. This proves Theorem 3.

§4. Description of the algebra of all derivations

Let $V = V(\mathfrak{A})$ be a classical homogeneous convex cone. We denote the Lie algebra of all its derivations by G. In §2 we constructed the subalgebra G^q of the algebra G, having the following two properties:

(A) the linear group generated by G^q acts transitively in V;

(B) every derivation of the cone V^c can be extended to a derivation of the cone V, belonging to G^q.

Let A be an arbitrary derivation of the cone V. By virtue of (A) it can be expressed in the form $A = \widetilde{A}_0 + \widetilde{A}_q$, where $\widetilde{A}_q \in G_q$, $\widetilde{A}_0 e = 0$. It follows from Theorem 3 that $\widetilde{A}_0 \mathfrak{X}^c \subset \mathfrak{X}^c$. The restriction of the transformation $\widetilde{A}_0$ to $\mathfrak{X}^c$ is a derivation of the cone V^c and, by property (B), can be extended to some transformation $\approx{A}_q \in G^q$. We put $A_q = \widetilde{A}_q + \approx{A}_q$, $A_0 = \widetilde{A}_0 - \approx{A}^q$. Then

$$A = A_0 + A_q \tag{6}$$

where $A_q \in G^q$, $A_0 \mathfrak{X}^c = 0$.

The decomposition (6) is unique. In fact, let A be an element in G^q such that $A\mathfrak{X}^c = 0$. Then, in particular, $Ae = 0$. In proving Proposition 5 we saw that this implies that $A \in G^c$; but then $A = 0$, since the representation of the Lie algebra G^c in the space $\mathfrak{X}^c$ is exact.

Thus it now suffices to find the derivations of the cone V which annihilate the space $\mathfrak{X}^c$. Clearly these derivations form a subalgebra in the Lie algebra G. We denote this subalgebra by G_0.

Proposition 6. *The linear Lie algebra* G_0 *coincides with the restriction to* $\mathfrak{X}$ *of the Lie algebra of the derivations of the T-algebra* $\mathfrak{A}$ *which become zero on the subspace* $\mathfrak{A}^c$.

Proof. Every derivation D of the algebra $\mathfrak{A}$ leaves the space of Hermitian matrices invariant and induces a transformation on it which is a derivation of the cone V. In fact, the transformations $C_\lambda = \exp \lambda D$ are automorphisms of the T-algebra $\mathfrak{A}$. For every triangular matrix $t \in \mathfrak{A}$ with positive elements on the main diagonal

$$C_\lambda(tt^*) = (C_\lambda t)\ (C_\lambda t)^* \in V.$$

This shows that C_λ induces a transformation on $\mathfrak{X}$ which is an automorphism of the cone V. Since $D = dC_\lambda / d\lambda|_{\lambda=0}$ the restriction of the transformation D to $\mathfrak{X}$ is a derivation of the cone V.

Now let D be a derivation of the cone V which becomes zero on $\mathfrak{X}^c$. Then, in particular, $De_i = 0$ for any i and D is a derivation of the connectedness algebra. For $i \neq j$ the subspace $\mathfrak{X}_{ij}$ may be characterized as follows:

$$\mathfrak{X}_{ij} = \left\{x \in \mathfrak{X}\colon e_i \,\square\, x = e_j \,\square\, x = \frac{1}{2} x\right\}.$$

This implies that $D\mathfrak{X}_{ij} \subset \mathfrak{X}_{ij}$. Let $x = x_{ij} + x_{ji}$ and $y = y_{jk} + y_{kj}$ be arbitrary elements in the spaces $\mathfrak{X}_{ij}$ and $\mathfrak{X}_{jk}$, respectively. If i, j, k are distinct, then

$$(x_{ij} + x_{ji}) \,\square\, (y_{jk} + y_{kj}) = \frac{1}{2} (x_{ij}y_{jk} + y_{kj}x_{ji}).$$

If we apply the derivation D to this equation and project onto $\mathfrak{A}_{ik}$, we find that

$$(Dx)_{ij}y_{jk} + x_{ij}(Dy)_{jk} = (D(x_{ij}y_{jk} + y_{kj}x_{ji}))_{ik}. \tag{7}$$

We extend the transformation D to the whole space $\mathfrak{A}$ by putting

$$Da_{ij} = (D(a_{ij} + a_{ij}^*))_{ij} \qquad (i \neq j).$$

Then (7) implies that

$$(Da_{ij})\, b_{jk} + a_{ij}\, (Db_{jk}) = D(a_{ij}b_{jk}), \tag{8}$$

for distinct i, j, k and also for $i = j$ and $j = k$, when it is trivially true.

We prove that the relation (8) also holds when $i = k$. We have that

$$(a_{ij} + a_{ij}^*) \,\square\, (b_{ij} + b_{ij}^*) = a_{ij}b_{ij}^* + a_{ij}^*b_{ij}.$$

Applying D and projecting onto $\mathfrak{A}_{ii}$ we find that

$$(Da_{ij})\, b^*_{ij} + a_{ij}(Db^*_{ij}) = 0,$$

which is what we had to prove. Thus D is a derivation of the T-algebra $\mathfrak{A}$. Since $D\mathfrak{X}^c = 0$, $D\mathfrak{A}^c = 0$. This proves the proposition.

If the derivation D of the T-algebra $\mathfrak{A}$ becomes zero on $\mathfrak{A}^c$ then it can be extended to the completion $\overline{\mathfrak{A}}$ of the algebra $\mathfrak{A}$ by means of the formulas

$$\overline{D}a = \begin{cases} 0 \text{ when } a \in \overline{\mathfrak{A}}^c, \\ Da \text{ when } a \in \mathfrak{A}^u. \end{cases}$$

It is not difficult to verify that D is a derivation of the matrix algebra $\overline{\mathfrak{A}}$. However, we shall need only the relations

$$\left.\begin{aligned} \overline{D}(a\cdot b) = a\cdot\overline{D}b, \\ \overline{D}(b\cdot a) = \overline{D}b\cdot a, \end{aligned}\right\} \quad (a \in \overline{\mathfrak{A}}^c,\ b \in \overline{\mathfrak{A}}). \tag{9}$$

They are obvious for $b \in \overline{\mathfrak{A}}^c$. The second relation is obtained from the first by involution. Therefore, it suffices to verify the first relation for $b \in \mathfrak{A}^u$. If $a \in \mathfrak{A}^c$ then by virtue of the formulas (7) of Chapter III,

$$\overline{D}(a\cdot b) = D(ab) = a(Db) = a(\overline{D}b) = a\cdot\overline{D}b.$$

Further, those a for which the first of relations (9) holds for all $b \in \mathfrak{A}^u$ form a subalgebra in $\overline{\mathfrak{A}}^c$. It is easy to see that $\mathfrak{A}^c$ generates $\overline{\mathfrak{A}}^c$ (with respect to multiplication denoted by a dot). Therefore the relations (9) hold for all $a \in \overline{\mathfrak{A}}^c$.

We show, by means of the formulas (9), that the derivations in G_0 commute with the transformations in G^c. Let $D \in G_0$, $a \in \mathfrak{D}$. Then

$$[D, D_a]\, x = D(a\cdot x + x\cdot a^*) - a\cdot Dx - Dx\cdot a^* = 0$$

for every $x \in \mathfrak{X}$.

Further, for $D \in G_0$ we see that for all $t \in \mathfrak{T}^u$, $x \in \mathfrak{X}$,

$$[D, D_t]\, x = D(tx + xt^*) - t(Dx) - (Dx)\, t^* = (Dt)\, x + x(Dt)^*$$

hence

$$[D, D_t] = D_{Dt}. \tag{10}$$

Summing up the results we have obtained, we may formulate the following theorem:

Theorem 4. *The Lie algebra of all the derivations of a classical homogeneous convex cone V splits into the direct sum*

$$G = G_0 + G^c + T^u,$$

where T^u is an ideal and G_0 and G^c are commuting subalgebras. Moreover, we have the following properties:

1) *the linear Lie algebra T^u is triangular;*

2) *the Lie algebra G^c is naturally isomorphic to the Lie algebra of derivations*

of the kernel V^c *of the cone* V.

3) *the Lie algebra* G_0 *is naturally isomorphic to the Lie algebra of those derivations of the T-algebra* $\mathfrak{A}$ *which annihilate its kernel* $\mathfrak{A}^c$;

4) *the subalgebra* $K \subset G$ *corresponding to the stationary subgroup of the point* $e \in V$ *splits into the direct sum*

$$K = G_0 + (K \cap G^c).$$

Remark 1. The term "naturally isomorphic" in the statement of the theorem means that the corresponding isomorphism is attained by the restriction of a linear transformation.

Remark 2. As a supplement to the theorem cf. the formulas (2), (4) and (10).

Remark 3. The theorem generalizes trivially to arbitrary homogeneous convex cones: for an exceptional cone we have only to take $G^c = G$.

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