# THE THEORY OF CONVEX HOMOGENEOUS CONES[1)]

*È. B. VINBERG*

## CONTENTS



## Introduction

A convex cone in a real finite-dimensional space $R$ is said to be *homogeneous* if the group of automorphisms of $R$ that leave the cone invariant acts transitively on it.

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1) The main results of this paper were presented at the meetings of the Moscow Mathematical Society on April 26, 1960 and April 10, 1962.

The need for studying convex homogeneous cones arose in the theory of automorphic functions on bounded homogeneous domains in $n$-dimensional complex space. In this theory it turned out to be convenient to consider unbounded models of bounded homogeneous domains, in which a certain group of complex affine transformations acts transitively. Models of this type are connected with convex homogeneous cones. If $V$ is a convex homogeneous cone in real $n$-dimensional space, then the domain

$$D = \{x + iy : y \in V\}$$

in $n$-dimensional complex space is analytically equivalent to a bounded homogeneous domain. This is the so-called "*generalized upper half-plane*" or "*Siegel domain of the first kind.*" It is homogeneous with respect to the group of complex affine transformations generated by the linear transformations

$$x + iy \longrightarrow Ax + iAy,$$

where $A$ is an automorphism of the cone $V$, and the translations

$$x + iy \longrightarrow (x + a) + iy.$$

Not all bounded homogeneous domains are equivalent to Siegel domains of the first kind. In 1957 I. I. Pjateckiĭ-Šapiro devised a more general construction, the so-called "*Siegel domains of the second kind*" (cf. [15], [18]), which are also connected with convex homogeneous cones. At the same time, at E. B. Dynkin's seminar on the theory of Lie groups, he posed the problem of classifying all convex homogeneous cones. (I. I. Pjateckiĭ-Šapiro's investigations on the theory of bounded homogeneous domains led, in particular, to the construction of examples of nonsymmetric domains [17]. In addition, it was proved that every bounded homogeneous domain is equivalent to a Siegel domain of the first or second kind. This was done in 1962 by I. I. Pjateckiĭ-Šapiro, S. G. Gindikin and the author of the present article (cf. [7]).)

Before 1960 apparently only the following convex homogeneous cones were known:

1) the cone of positive-definite symmetric matrices;

2) the cone of positive-definite Hermitian complex matrices;

3) the cone of positive-definite Hermitian quaternion matrices;

4) the 27-dimensional cone of, in a certain sense, "positive-definite" Hermitian octavic matrices of third order;

5) the spherical cone

$$x_0 > \sqrt{x_1^2 + \cdots + x_n^2}.$$

All these cones are selfadjoint in the appropriate Euclidean metric. To each of them there corresponds one of Cartan's symmetric domains [9]. In 1960 the author succeeded in constructing a large number of new examples of convex homogeneous cones and in establishing a one-to-one correspondence between all the convex homogeneous cones and nonassociative algebras of a special form, the so-called compact left-symmetric algebras, or clans [3], [5]. (A similar algebraic apparatus was proposed by Koszul [13].) Pjateckiĭ-Šapiro's results on the classification of bounded homogeneous domains [19]–[21] made it possible to classify compact left-symmetric algebras as well. This was done by B. Ju. Veĭsfeĭler and the author. However, it should be noted that this classification does not have the definite nature of, say, the classification of semisimple Lie algebras, and it opens up the possibility of further investigations in different directions.

In the present article we shall use the classification of compact left-symmetric algebras as a basis for constructing the apparatus of generalized matrix algebras, the so-called $T$-algebras, which will allow us to consider any convex homogeneous cone as a cone of "positive definite Hermitian matrices." This is the most substantial part of the whole work.

The apparatus of $T$-algebras is well adapted for the description of convex homogeneous cones and for the solution of problems of different kinds. In particular, it has been used successfully to find the complete group of automorphisms of an arbitrary convex homogeneous cone. This result was formulated in general terms in the note [6] and will soon be published in full.

By the use of $T$-algebras it is not difficult to classify all the selfadjoint homogeneous cones. Each such cone splits into the direct sum of cones of the five types listed above. We observe, however, that the selfadjoint homogeneous cones can be described by an independent theory of their own, based on their connection with semisimple Jordan algebras (Köcher [11], [12], Vinberg [3], Hertneck [24]).

M. Köcher introduced a special kind of convex cone, the domains of positivity [10]. These are convex cones that are selfadjoint in some Euclidean or pseudo-Euclidean metric, which is called the *characteristic*. All the theorems proved by Köcher [10] and Rothaus [22] for domains of positivity carry over without any difficulty to arbitrary convex cones (cf. [2], and also Chapter I of the present article). Apparently it is not worth while to single out the domains of positivity from all the convex cones. The following considerations argue to this end. If $V$ is a convex cone and $V'$ is its adjoint cone, then it is easy to see that the direct sum $V + V'$ is a domain of positivity, which is homogeneous if the cone $V$ is homogeneous. Therefore, it is unlikely that the domains of positivity can have some remarkable properties that are not shared by arbitrary convex cones. Of course, I am not referring here to domains of positivity with a positive-definite characteristic.

We shall now give a brief account of the contents of the article.

Chapter I is introductory. In it we give the fundamental definitions and construct the geometry of convex cones (basically following Köcher and Rothaus, cf. above). In addition, we prove some of the simplest properties of the group of automorphism $\mathfrak{G}(V)$ of a convex homogeneous cone $V$, in particular, the following: the stationary subgroup of an arbitrary point $x \in V$ is a maximal compact subgroup in $\mathfrak{G}(V)$; every maximal connected triangular subgroup of the group $\mathfrak{G}(V)$ acts simply transitively on $V$.

In Chapter II we establish the one-to-one correspondence between $n$-dimensional convex homogeneous cones and $n$-dimensional algebras over the real field that satisfy the following conditions (the sign $\Delta$ denotes multiplication in these algebras):

1) $a\Delta(b\Delta c) - (a\Delta b)\Delta c = b\Delta(a\Delta c) - (b\Delta a)\Delta c$ (left-symmetry);

2) there exists a linear form $s$ such that $s(a\Delta b) = s(b\Delta a)$ and $s(a\Delta a) > 0$ for $a \neq 0$ (compactness);

3) the eigenvalues of the operators $L_a : x \to a\Delta x$ are real;

4) there exists a unit element $e$: $e\Delta a = a\Delta e = a$.

Such algebras are called *clans with a unity element.* Using them we show that every convex homogeneous cone $V$ can be regarded as the set of points $(t, x, u) \in R^1 \times R^k \times R^m$ satisfying the equations

$$tx - F(u, u) \in V_0,$$

$$t > 0,$$

where $V_0$ is a convex homogeneous cone in $R^k$ and $F(u, v)$ is a bilinear form on $R^m \times R^m$ with values in $R^k$. Finally, we give the classification of clans which forms the basis for Chapter III.

In Chapter III we construct the apparatus of *T-algebras.* In view of the importance of this concept we shall define the $T$-algebras here. (This definition differs in form from that given in §1 of Chapter III.)

We consider the square matrices $A = (a_{ij})$ whose elements belong to arbitrary vector spaces:

$$a_{ij} \in \mathfrak{A}_{ij}.$$

We assume that for every triplet $i, j, k$ there is defined a bilinear mapping

$$\mathfrak{A}_{ij} \times \mathfrak{A}_{jk} \to \mathfrak{A}_{ik}.$$

This can be considered as multiplication of matrix elements. For any matrices $A = (a_{ij})$ and $B = (b_{ij})$ we can then define the product $AB = (c_{ij})$ in the usual way, putting

$$c_{ij} = \sum_k a_{ik} b_{kj}.$$

Suppose, in addition, that to every pair $i, j$ there is given an isomorphic correspondence $a_{ij} \longleftrightarrow \overline{a}_{ij}$ between the spaces $\mathfrak{A}_{ij}$ and $\mathfrak{A}_{ji}$, satisfying the conditions

1) $\overline{\overline{a}}_{ij} = a_{ij}$;

2) $\overline{a_{ij}b_{jk}} = \overline{b}_{jk}\overline{a}_{ij}$.

Then for every matrix $A = (a_{ij})$ we can define the "adjoint" matrix $A^* = (b_{ij})$ by the formula

$$b_{ij} = \overline{a}_{ji}.$$

The mapping $A \longrightarrow A^*$ has the usual properties. Thus, we have a meaning for such concepts as "Hermitian" matrix, etc.

The space $\mathfrak{A}$ of generalized matrices, with the multiplication and involution $A \longrightarrow A^*$ considered above, will be called a *matrix algebra with involution.* The classical examples of matrix algebras with involution are the algebras of real, complex and quaternion matrices. All these algebras are associative, but we shall not restrict ourselves to associative matrix algebras.

A matrix algebra $\mathfrak{A}$ with involution is called a *T-algebra* if the following conditions are satisfied:

1) *for every* $i$ *the subalgebra* $\mathfrak{A}_{ii}$ *is isomorphic to the algebra of real numbers;*

2) *for every* $a_{ij} \in \mathfrak{A}_{ij}$, $a_{ij} \neq 0$,

$$a_{ij}\overline{a}_{ij} > 0;$$

3) *the associativity relation*

$$a_{ij}(b_{jk}c_{kl}) = (a_{ij}b_{jk})c_{kl}$$

*holds for all* $a_{ij} \in \mathfrak{A}_{ij}$, $b_{jk} \in \mathfrak{A}_{jk}$, $c_{kl} \in \mathfrak{A}_{kl}$ *if* $i \neq k$, $j \neq l$ *and the pair of indices* $i, k$ *does not differ (strictly) from the pair* $j, l$ *by a point on the real line;*

4) *if* $j$ *lies between* $i$ *and* $k$, *then*

$$\overline{a}_{ij}(a_{ij}b_{jk}) = (\overline{a}_{ij}a_{ij})b_{jk}$$

*for any* $a_{ij} \in \mathfrak{A}_{ij}$, $b_{jk} \in \mathfrak{A}_{jk}$;

5) *there exist positive numbers* $n_i$ *such that*

$$n_i a_{ij}\overline{b}_{ij} = n_j \overline{b}_{ij}a_{ij}$$

*for any* $a_{ij}, b_{ij} \in \mathfrak{A}_{ij}$.

If $\mathfrak{A}$ is an arbitrary $T$-algebra, then the set $V(\mathfrak{A})$ consisting of those Hermitian matrices of the algebra $\mathfrak{A}$ that can be expressed in the form $TT^*$, where $T$ is a triangular matrix with positive elements on the diagonal, is a convex homogeneous cone. We shall show in Chapter III that we can obtain all convex homogeneous cones in this way, where isomorphic cones result only from isomorphic

$T$-algebras (cf. Definition 5 in Chapter III).

The $T$-algebra $\mathfrak{A}$ is convenient for describing the simply transitive group of automorphisms of the cone $V(\mathfrak{A})$, its geometry and its adjoint cone. Also it enables us to write out explicitly the inequalities defining the cone $V(\mathfrak{A})$. All this is done in Chapter III.

The indecomposable selfadjoint cones correspond exactly to those $T$-algebras in which all the subspaces $\mathfrak{A}_{ij}$, $i \neq j$, have exactly the same dimension.

Throughout the following we shall use the following notation without comment: if $M$ is a set in a topological space, then its closure will be denoted by $\overline{M}$.

Definition, lemmas, propositions and formulae are numbered separately in each chapter; references to them within a chapter are given without the number of the chapter being specified. Theorems are numbered consecutively throughout.

In conclusion I should like to express my deep gratitude to my scientific director E. B. Dynkin for his invaluable help in the writing of this article. I should like to express my sincere gratitude to I. I. Pjateckiĭ-Šapiro, with whom I had many useful discussions and who constantly kept me informed about his investigations on the classification of bounded homogeneous domains. I should also like to thank the participants in E. B. Dynkin's seminar on the theory of Lie groups, and in particular S. G. Gindikin, for their interest in my work.

# CHAPTER I

# GENERAL PROPERTIES AND GEOMETRY OF CONVEX CONES

## §1. Introduction

The main object of study in this article will be the open strictly convex cones in finite-dimensional real linear spaces. For simplicity we shall refer to them as *convex cones*.

Thus, departing a little from the usual terminology, we adopt the following

Definition 1. A *convex cone in a finite-dimensional real space* $R$ is a nonempty set $V \subset R$ having the following properties:

(C1) if $x \in V$ and $\lambda$ is a positive number, then $\lambda x \in V$;

(C2) if $x, y \in V$, then $x + y \in V$;

(C3) the set $V$ does not completely contain any straight line (not necessarily passing through 0);

(C4) the set $V$ is open in $R$.

The condition (C3) is equivalent to the fact that the closure of $V$ does not contain a subspace of positive dimension.

The convex cones $V_1 \subset R_1$ and $V_2 \subset R_2$ are considered to be *isomorphic* if there exists an isomorphism of the linear spaces $R_1$ and $R_2$ under which the sets $V_1$ and $V_2$ correspond to one another.

For every linear space $R$ we shall denote by $R'$ the space of linear functionals on $R$ and by $\langle x, x'\rangle$ or $\langle x', x\rangle$ the value of the linear functional $x' \in R'$ at the vector $x \in R$.

**Definition 2.** Let $V$ be a convex cone in the space $R$. The *convex cone adjoint to* $V$ is the cone $V'$ in the space $R'$ that is formed by $x' \in R'$ for which $\langle x, x'\rangle > 0$ for all $x \in \overline{V}$, $x \neq 0$.

It is well known that $V'' = V$ (under the natural identification of the spaces $R''$ and $R$; cf. for example [1]).

**Definition 3.** *The direct sum of a finite set of convex cones* $V_i \subset R_i$ is the convex cone $\Sigma V_i$ in the direct sum $\Sigma R_i$ of the linear spaces $R_i$ that is formed by those vectors $\Sigma x_i \in \Sigma R_i$ $(x_i \in R_i)$ for which $x_i \in V_i$ for all $i$.

Clearly $(\Sigma V_i)' = \Sigma V'_i$ (under the natural identification of the spaces $(\Sigma R_i)'$ and $\Sigma R'_i$). It is not difficult to prove

**Proposition 1.** *Every convex cone splits in a unique way into the direct sum of indecomposable convex cones.*

The nondegenerate linear transformations $A$ of the space $R$ that leave the convex cone $V$ invariant in the sense that $AV = V$ are naturally called the *automorphisms* of $V$. They form a group, which will be denoted by $\mathfrak{G}(V)$. It is easy to see that the group $\mathfrak{G}(V)$ is closed in the complete linear group. Its Lie algebra will be denoted by $G(V)$ and the linear transformations in $G(V)$ will be called *derivations* of $V$.

It follows from the invariance of the definition of the adjoint cone that if $A$ is an automorphism of $V$, then the adjoint operator $A'$ is an automorphism of $V'$. The mapping $A \rightarrow A'^{-1}$ induces an isomorphism of $\mathfrak{G}(V)$ onto $\mathfrak{G}(V')$.

Suppose that $V = \Sigma V_i$ (cf. Definition 3) and that for each $i$ we are given an automorphism $A_i$ of $V_i$. The operator $\Sigma A_i$, transforming $\Sigma x_i$ into $\Sigma A_i x_i$ is then an automorphism of $V$. The mapping $(A_i) \rightarrow \Sigma A_i$ induces an imbedding of the direct product $\Pi \mathfrak{G}(V_i)$ into $\mathfrak{G}(V)$. If the cones $V_i$ are indecomposable, then, by Proposition 1, the automorphisms of $V$ can only permute the components $V_i$ among themselves. Hence we obtain

**Proposition 2.** *If* $V = \Sigma V_i$ *is the decomposition of a convex cone* $V$ *into indecomposable cones* $V_i$, *then* $\Pi \mathfrak{G}(V_i)$ *is a normal subgroup of finite index in* $\mathfrak{G}(V)$.

**Definition 4.** A convex cone will be said to be *homogeneous* if the group $\mathfrak{G}(V)$ acts transitively on it.

Clearly, the direct sum of convex homogeneous cones is also a convex homogeneous cone. The homogeneity of the adjoint cone will be proved in §4.

## §2. The characteristic function

Let $V$ be a convex cone in the $n$-dimensional linear space $R$ and let $dx'$ be a measure in $R'$ that is invariant under parallel translations. We put

$$\varphi_V(x) = \varphi(x) = \int_{V'} e^{-\langle x, x'\rangle} dx' \quad (1)$$

for every $x \in V$. We shall prove that this integral converges. Let $dx'_{(1)}$ be the measure on the hyperplanes $P_\alpha = \{x' \colon \langle x, x'\rangle = \alpha\}$ of the space $R'$, defined by the following condition: for any continuous function $f$ of compact support on $R'$

$$\int_{R'} f(x')\, dx' = \int_{-\infty}^{+\infty} d\alpha \int_{P_\alpha} f(x')\, dx'_{(1)}.$$

Then

$$\varphi(x) = \int_0^\infty e^{-\alpha}\, d\alpha \int_{P_\alpha \cap V'} dx'_{(1)}.$$

Since $x \in V = V''$, $\langle x, x'\rangle > 0$ for every $x' \in \overline{V}'$, $x' \neq 0$. Therefore, every ray $\{\lambda x'\}$, where $x' \in \overline{V}'$, $x' \neq 0$, meets the hyperplane $P_\alpha$, so the cross-section of the cone $\overline{V}'$ by the hyperplane $P_\alpha$ is bounded and the volume $v(\alpha)$ of the set $P_\alpha \cap V' \subset P_\alpha$ is finite. Since the hyperplane $P_\alpha$ is obtained from $P_1$ by a similarity transformation with coefficient $\alpha$,

$$v(\alpha) = \alpha^{n-1} v(1).$$

Finally,

$$\varphi(x) = v(1) \int_0^\infty e^{-\alpha} \alpha^{n-1}\, d\alpha = (n-1)!\, v(1) < \infty.$$

In the integral (1) the exponential may be replaced by any sufficiently rapidly decreasing function of $\langle x, x'\rangle$. An argument similar to the above shows that in this case the integral is multiplied by a quantity independent of $x$. More precisely,

$$\int_{V'} F(\langle x, x'\rangle)\, dx' = \frac{\int_0^\infty F(\alpha)\, \alpha^{n-1}\, d\alpha}{(n-1)!}\, \varphi(x).$$

**Definition 5.** The function $\phi_V$ defined on the convex cone $V$ by the formula (1) is called the *characteristic function of* $V$.

The characteristic function is defined to within a constant positive factor, because this degree of arbitrariness is present in the definition of the measure $dx'$.

**Proposition 3.** *The characteristic function grows without limit on approach to* *any point on the boundary of $V$.*

**Proof.** Suppose that the sequence of points $x_k$ $(k = 1, 2, \cdots)$ of $V$ converges to the boundary point $x_0$. The functions $F_k(x') = e^{-\langle x_k, x'\rangle}$ are nonnegative and converge to $F_0(x') = e^{-\langle x_0, x'\rangle}$ uniformly on any bounded set in the space $R'$. Therefore,

$$\underline{\lim}_{k\to\infty} \varphi(x_k) = \underline{\lim}_{k\to\infty} \int_{V'} F_k(x')\, dx' \geqslant \int_{V'} F_0(x')\, dx',$$

so it is sufficient to show that the integral $\int_{V'} F_0(x')dx'$ diverges. There exists a vector $x'_0 \in \overline{V}'$, $x'_0 \neq 0$, such that $\langle x_0, x'_0\rangle = 0$. In fact, for the vector $x'_0$ we may take the linear functional defined by the hyperplane of support of $V$ passing through $x_0$. Further, we take a closed ball $K$ lying entirely within $V'$ and consider the set

$$L = K + \{\lambda x'_0\}_{\lambda > 0} \subset V'.$$

Let $c = \min_{x' \in K} F_0(x')$. Clearly, $c > 0$ and $c = \min_{x' \in L} F_0(x')$. We have

$$\int_{V'} F_0(x')\, dx' \geqslant \int_L F_0(x')\, dx' \geqslant c \int_L dx' = \infty,$$

as required.

Proposition 3 shows that the function $\phi_V$ really does characterize the cone $V$. For results of the following kind are valid: *if $V_1$ and $V_2$ are open convex cones in the space $R$, $V_1 \cap V_2 \neq \emptyset$, and $\phi_{V_1} = \phi_{V_2}$ on $V_1 \cap V_2$, then $V_1 = V_2$.* For, if under these assumptions the cone $V_1$, for example, were to lie partly outside $V_2$, then there would exist a point $x_0 \in R$ interior to $V_1$ and a boundary point of $V_2$, and the equation $\phi_{V_1}(x_0) = \phi_{V_2}(x_0)$ would not be possible, because by what has been proved above, $\phi_{V_2}(x_0) = \infty$, whereas $\phi_{V_1}(x_0) < \infty$.

Let us clarify how the characteristic function behaves under the automorphisms. Let $A$ be an automorphism of $V$; then

$$\varphi(Ax) = \int_{V'} e^{-\langle Ax,\, x'\rangle}\, dx' = \int_{V'} e^{-\langle x,\, A'x'\rangle}\, dx'.$$

If we make the change of variables $y' = A'x'$, $y' \in V'$, we find that

$$\varphi(Ax) = \frac{\varphi(x)}{\det A}. \tag{2}$$

This immediately implies

**Proposition 4.** *The measure $\phi(x)dx$ is invariant under all the automorphisms of $V$.*

The characteristic function of a convex cone has a remarkable differential property.

We recall that a continuous function $f$ defined on a convex subset $M$ of an affine space is said to be (*strictly*) *convex* if, for any $x_1, x_2 \in M$ and for any point $x$ lying on the interval joining $x_1$ and $x_2$ and dividing it in the ratio $p:q$ $(p+q=1;\ p,\ q>0)$,

$$f(x) < pf(x_1) + qf(x_2).$$

If the function $f$ is twice differentiable and the set $M$ is open, then a necessary and sufficient condition for $f$ to be convex is that the quadratic form $d^2 f$ should be positive-definite at all the points of $M$ (cf., for example, [23], Chapter III).

Proposition 5. *Let $V$ be an open convex cone in the linear space $R$. The function* $\ln \phi_V$ *is a convex function on* $V$.

Proof. Clearly

$$d \ln \varphi = \frac{d\varphi}{\varphi}, \qquad d^2 \ln \varphi = \frac{d^2\varphi}{\varphi} - \left(\frac{d\varphi}{\varphi}\right)^2.$$

If we differentiate the integral (1) with respect to $x$, we obtain the following expressions for the differentials of $\phi_V = \phi$ at $x \in V$:

$$(d\varphi(x))(a) = -\int_{V'} e^{-\langle x, x'\rangle} \langle a, x'\rangle \, dx',$$

$$(d^2\varphi(x))(a) = \int_{V'} e^{-\langle x, x'\rangle} \langle a, x'\rangle^2 \, dx',$$

where $a$ is an arbitrary vector in $R$.

We put

$$F(x') = e^{-\frac{1}{2}\langle x, x'\rangle}, \qquad G(x') = e^{-\frac{1}{2}\langle x, x'\rangle} \langle a, x'\rangle.$$

Then, for $a \neq 0$,

$$(d^2 \ln \varphi(x))(a) = \frac{1}{(\varphi(a))^2} \left[ \int_{V'} F^2 \, dx' \cdot \int_{V'} G^2 \, dx' - \left( \int_{V'} FG \, dx' \right)^2 \right] > 0,$$

since the functions $F$ and $G$ are not proportional.

We observe that not only $\ln \phi$ but also $\phi$ itself is convex, because

$$d^2\varphi = \varphi d^2 \ln \varphi + \frac{1}{\varphi}(d\varphi)^2 > 0.$$

However, there are reasons for which it is preferable to consider $\ln \phi$ rather than $\phi$ itself. One of these reasons is the fact that, while $\phi$ is determined to within a constant factor, $\ln \phi$ is determined to within an additive constant and its differentials are uniquely determined. This circumstance will be important in what follows.

## §3. The canonical Riemannian geometry

We shall regard the cone $V$ as a differentiable manifold. By means of a parallel translation we shall identify the tangent space to $V$ at any of its points with the containing linear space $R$.

The quadratic differential form on $V$ that coincides with $d^2 \ln \phi$ in any affine coordinate system defines the structure of a *Riemannian space* on $V$. In an affine coordinate system the components of the *metric tensor* $g$ are given by the formula

$$g_{ij} = \partial_{ij} \ln \phi. \tag{3}$$

We denote by $\Gamma$ the object of the *canonical torsion-free linear connection* defined by the Riemannian metric $g$; it is well known that

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} (\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}),$$

where $g^{il}$ is the tensor inverse to $g_{ij}$, i.e., $g^{ij} g_{jk} = \delta^i_k$. In our case it follows from (3) that in any affine coordinate system

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \partial_{jkl} \ln \varphi. \tag{4}$$

The object $\Gamma$ is not a tensor, but its components transform like the components of a tensor under affine coordinate transformations.

Let $x_0$ be any point in $V$. We identify the space $R$ with the tangent space to $V$ at $x_0$ and in it we define an operation of multiplication $\square$, where in any affine coordinate system we put

$$(a \,\square\, b)^i = -\Gamma^i_{jk}(x_0) a^j b^k \qquad (a, b \in R). \tag{5}$$

This multiplication is *commutative*, since the connection $\Gamma$ is torsion-free, i.e., $\Gamma^i_{jk} = \Gamma^i_{kj}$.

**Definition 6.** Let $V$ be a convex cone in a linear space $R$ and let $x_0 \in V$. The space $R$, equipped with the operation of multiplication $\square$ by the formula (5) is called the *algebra of connectedness of* $V$ at $x_0$.

It follows from (2) that for any automorphism $A$ of $V$

$$\ln \phi(Ax) = \ln \phi(x) - \ln \det A.$$

Therefore, the linear differential form $d \ln \phi$ and the quadratic differential form $d^2 \ln \phi$ are invariant under the automorphism $A$ of $V$.

**Proposition 6.** *The canonical Riemannian geometry defined on the convex cone* $V$ *by the quadratic differential form* $d^2 \ln \phi$ *is invariant under all its automorphisms. Every automorphism of the cone* $V$ *that leaves the point* $x_0 \in V$ *fixed is an automorphism of the algebra of connectedness of* $V$ *at* $x_0$.

## §4. The correspondence between the points of adjoint cones

Let $V$ be a convex cone in an $n$-dimensional linear space $R$ and let $\phi$ be its characteristic function. If $x$ is any point of $V$ and the tangent space to $V$ at $x$ is identified with $R$, then the differential $d\ln\phi(x)$ of the function $\ln\phi$ at the point $x$ can be considered to be an element of $R'$. We put

$$x^* = -d\ln\varphi(x) = -\frac{d\varphi(x)}{\varphi(x)} \in R'. \tag{6}$$

Let us examine the geometrical significance of the mapping $^*$. If we differentiate the formula (1), we find that

$$d\varphi(x) = -\int\limits_{V'} e^{-\langle x, x'\rangle} x'\, dx'$$

(where a vector function stands under the integral sign). Therefore,

$$x^* = \frac{\int\limits_{V'} e^{-\langle x, x'\rangle} x'\, dx'}{\int\limits_{V'} e^{-\langle x, x'\rangle}\, dx'}.$$

If we use the same notation as at the beginning of §2, we find that

$$x^* = \frac{\int\limits_0^\infty e^{-\alpha}\, d\alpha \int\limits_{P_\alpha \cap V'} x'\, dx'_{(1)}}{\int\limits_0^\infty e^{-\alpha}\, d\alpha \int\limits_{P_\alpha \cap V'} dx'_{(1)}}.$$

Clearly

$$\int\limits_{P_\alpha \cap V'} x'\, dx'_{(1)} = \left(\frac{\alpha}{n}\right)^n \int\limits_{P_n \cap V'} x'\, dx'_{(1)},$$

$$\int\limits_{P_\alpha \cap V'} dx'_{(1)} = \left(\frac{\alpha}{n}\right)^{n-1} \int\limits_{P_n \cap V'} dx'_{(1)}.$$

Therefore

$$x^* = \frac{\int\limits_0^\infty e^{-\alpha}\alpha^n\, d\alpha \cdot \int\limits_{P_n \cap V'} x'\, dx'_{(1)}}{n\int\limits_0^\infty e^{-\alpha}\alpha^{n-1}\, d\alpha \cdot \int\limits_{P_n \cap V'} dx'_{(1)}} = \frac{\int\limits_{P_n \cap V'} x'\, dx'_{(1)}}{\int\limits_{P_n \cap V'} dx'_{(1)}},$$

i.e., $x^*$ *is the center of gravity of the cross-section of* $V'$ *by the hyperplane* $P_n = \{x' : \langle x, x'\rangle = n\}$. In particular, $x^* \in V'$ and

$$\langle x, x^*\rangle = n. \tag{7}$$

Moreover, this implies that

$$(\lambda x)^* = \frac{x^*}{\lambda} \tag{8}$$

for every $\lambda > 0$.

There is another possible geometrical interpretation of the mapping *. We consider the hypersurface

$$S_x = \{y \in V: \varphi(y) = \varphi(x)\}$$

in the space $R$. We denote by $Q_x$ the tangent hyperplane to $S_x$ at $x$. Clearly

$$Q_x = \{z \in R: (d\varphi(x))(z - x) = 0\} = \{z \in R: \langle x^*, z\rangle = \langle x^*, x\rangle\}.$$

Finally, using (7) we find that

$$Q_x = \{z \in R: \langle x^*, z\rangle = n\}.$$

This condition can be taken as the definition of the element $x^* \in R'$.

**Proposition 7.** *The mapping $x \longrightarrow x^*$ sets up a one-to-one correspondence between the points of the cones $V$ and $V'$.*

**Proof.** We showed above that $x^* \in V'$; it remains to show that for every $x' \in V'$ there exists a unique element $x \in V$ for which $x^* = x'$.

Let $x' \in V'$. We consider the cross-section of $V$ by the hyperplane

$$Q = \{z \in R: \langle x', z\rangle = n\}.$$

If there exists a point $x \in V$ such that $x^* = x'$, then it lies on the hyperplane $Q$ (cf. (7)) and for every $z \in Q$

$$(d \ln \varphi(x))(z - x) = \langle x^*, x - z\rangle = n - \langle x', z\rangle = 0;$$

since $d^2 \ln \phi(x) > 0$, this means that the restriction of $\phi$ to $Q \cap V$ attains a minimum (at least, locally) at $x$. On the other hand, if the restriction of $\phi$ to $Q \cap V$ attains a minimum at $x$, then, for every $z \in Q$,

$$0 = (d \ln \varphi(x))(z - x) = -\langle x^*, z\rangle + n$$

and $x^* = x'$.

Thus, the problem reduces to showing that $\phi$, considered on $Q \cap V$, has a unique minimum. By Proposition 3, on approaching any boundary point of the set $Q \cap V \subset Q$ the function $\phi$ tends to $+\infty$. Since $Q \cap V$ is bounded, this implies that $\phi$ has a minimum on $Q \cap V$. Let us prove that it is unique. Let $x_1, x_2 \in Q \cap V$ be two minimum points. Since $\phi$ is convex (cf. the remark on Proposition 5), the value of $\phi$ at all interior points of the interval connecting $x_1$ and $x_2$ is less than $\max\{\phi(x_1), \phi(x_2)\}$; this follows at once from the definition of a convex function. Let us assume for definiteness that $\phi(x_1) \geq \phi(x_2)$; then $x_1$ cannot be a minimum, even locally.

**Ptoposition 8.** *If $A$ is an automorphism of the convex cone $V$, then for every $x \in V$*

$$(Ax)^* = A'^{-1} x^*.$$

Proof. It follows from the invariance of the linear differential form $d \ln \phi$ (cf. §3) that for any $a \in R$

$$(d \ln \phi(Ax))(Aa) = (d \ln \phi(x))(a),$$

which can be rewritten as follows:

$$\langle (Ax)^*, Aa \rangle = \langle x^*, a \rangle.$$

This is clearly equivalent to Proposition 8.

Proposition 9. *The convex cone $V'$ adjoint to a convex homogeneous cone $V$ is homogeneous. Moreover, if the group $\mathfrak{G} \subset \mathfrak{G}(V)$ acts transitively on $V$, then the group $\mathfrak{G}' = \{A' : A \in \mathfrak{G}\} \subset \mathfrak{G}(V')$ acts transitively on $V'$.*

Proof. Let $x', y' \in V'$. By Proposition 7, there exists vectors $x, y \in V$ such that $x' = x^*$, $y' = y^*$. Further, there exists a transformation $A \in \mathfrak{G}$ such that $Ax = y$. Thus

$$A'y' = A'y^* = A'(Ax)^* = x^* = x'$$

(cf. Proposition 8). Since $x'$, $y'$ are arbitrary points of $V'$, this means that $\mathfrak{G}'$ acts transitively on $V'$.

If $V$ is homogeneous, then the mapping $^*$ has several simple properties which it does not have in the general case. First of all,

$$\phi_V(x)\phi_{V'}(x^*) = \text{const}\,(x \in V). \tag{9}$$

In fact, for every $x \in V$ and for every automorphism $A$ of $V$

$$\varphi_V(Ax)\,\varphi_{V'}((Ax)^*) = \varphi_V(Ax)\,\varphi_{V'}((A')^{-1}x^*)$$
$$= \frac{\varphi_V(x)\,\varphi_{V'}(x^*)}{\det A \cdot \det (A')^{-1}} = \varphi_V(x)\,\varphi_{V'}(x^*).$$

Since $V$ is homogeneous, this implies (9).

In the same way as we defined a mapping $^*$ of $V$ onto $V'$ we can define a mapping of $V'$ onto $V$, inasmuch as $V'' = V$. We shall denote this mapping by the same symbol $^*$.

Proposition 10. *Let $V \subset R$ be a convex homogeneous cone. For every $x \in V$*

$$x^{**} = x.$$

Proof. Let $g$ be the tensor (more precisely, the tensor field) of the canonical Riemannian metric on $V$. For $x \in V$ the tensor $g(x)$ can be regarded as a symmetric bilinear form on $R$. It follows directly from the definitions (cf. formulae (3) and (6)) that for any $a, b \in R$

$$(d \langle x^*, a \rangle)(b) = -(g(x))(a, b), \tag{10}$$

where the differential on the left-hand side is taken with respect to $x$. From (9) we find that

$$\ln \varphi_V(x) + \ln \varphi_{V'}(x^*) = \text{const} \quad (x \in V).$$

Differentiating this relation with respect to $x$ we find that (cf. (10))

$$-\langle x^*, a\rangle + (g(x))(x^{**}, a) = 0 \tag{11}$$

for every $a \in R$.

On the other hand, if we differentiate the relation (7) with respect to $x$ we find that (cf. (10))

$$\langle x^*, a\rangle - (g(x))(x, a) = 0 \tag{12}$$

for every $a \in R$. It follows from (11) and (12) that for all $a \in R$

$$(g(x))(x^{**}, a) = (g(x))(x, a), \tag{13}$$

whence $x^{**} = x$.

Proposition 10 has an interesting geometrical interpretation. Let $x \in V$ and let $S_x$ be the level surface of the characteristic function of $V$ passing through $x$. The hyperplane

$$Q_x = \{z \in R : \langle x^*, z\rangle = n\}, \tag{14}$$

where $n$ is the dimension of $R$, passes through $x$ and is tangent to $S_x$ at this point (cf. above). On the other hand, the center of gravity of the set $Q_x \cap V \subset Q_x$ coincides with $x^{**}$, and therefore with $x$ if $V$ is homogeneous. Thus, the convex homogeneous cones have the following remarkable property: *every hyperplane cross-section is tangent at its center of gravity to the level surface of the characteristic function.* We observe that this property does not hold, in general, for convex nonhomogeneous cones, as is shown by the example of a tetrahedral angle in three-dimensional space.

## §5. Convex homogeneous domains

In §6 of the present chapter and also in Chapter II it will be natural to consider more general objects than homogeneous cones. We shall adopt the following

Definition 7. A *convex domain* in an affine space $P$ is any nonempty open convex set $U \subset P$ not completely containing any straight line.

The convex domains $U_1 \subset P_1$ and $U_2 \subset P_2$ are regarded as *isomorphic* if there exists an isomorphism of the affine spaces $P_1$ and $P_2$ under which $U_1$ and $U_2$ correspond to one another.

Clearly, a convex cone is a special case of a convex domain. The vertex of the cone defines a "center" in the affine space and converts it into a linear space.

The affine transformations of the space $P$ that leave $U$ invariant are called its *automorphisms*. They form a group which will be denoted by $\mathfrak{G}(U)$. It is easy to see that $\mathfrak{G}(U)$ is closed in the group of all nondegenerate affine transformations of the space $P$. If $U$ is a cone with vertex $o$, then all the automorphisms of it leave the point $o$ invariant and are consequently linear transformations of the linear space that is obtained from $P$ by taking the point $o$ as center.

We shall denote the Lie algebra of $\mathfrak{G}(U)$ by $G(U)$ and refer to its elements as *derivations* of $U$. The elements of $G(U)$ are "*infinitesimal affine transformations.*" We need a digression here to explain how this term is to be understood and to establish the rules of operating with infinitesimal affine transformations.

Let $P$ be an affine space. We denote the linear space of free vectors of $P$ by $R_P$. If we consider $P$ as a differentiable manifold, then the tangent space to it at any point can be identified in a natural way with the space $R_P$. For $x \in P$, $a \in R_P$ we agree to use $x + a$ to denote the point of $P$ that is the end of the vector $a$ when its initial point is transferred to $x$; for $x, y \in P$, $x - y$ will denote the element of $R_P$ that can be represented by the vector with initial point at $y$ and end point at $x$.

If we now choose some point $x_0$ in $P$, then every affine transformation $C$ can be written in the form

$$Cx = \dot{C}(x - x_0) + c \tag{15}$$

where $c \in P$ and $\dot{C}$ is a linear transformation of $R_P$ that does not depend on the choice of $x_0$ and is called the *linear part of the transformation* $C$. An infinitesimal affine transformation may be thought of as a vector field on $P$, namely the velocity field of some one-parameter group of affine transformations. In other words, every infinitesimal affine transformation $D$ is a mapping of $P$ into $R_P$. It can be written in the form

$$D(x) = A(x - x_0) + a, \tag{16}$$

where $a \in R_P$ and $A$ is a linear transformation of $R_P$ that does not depend on the choice of $x_0$ and is called the *linear part of the infinitesimal transformation* $D$.

It is easy to derive the following *commutation law* for infinitesimal affine transformations:

$$[D_1, D_2](x) = A_1 D_2(x) - A_2 D_1(x), \tag{17}$$

where $A_i$ $(i = 1, 2, )$ is the linear part of the transformation $D_i$. The *exponential mapping* of the Lie algebra of infinitesimal affine transformations into the group of nondegenerate affine transformations is given by the formula

$$(\exp D)\, x = x + \sum_{k=0}^{\infty} \frac{A^k}{(k+1)!} D(x), \tag{18}$$

where $A$ is the linear part of $D$.

**Definition 8.** The convex domain $U$ is said to be *homogeneous* if the group $\mathfrak{G}(U)$ acts on it transitively.

We observe for later use that if some group $\mathfrak{G} \subset \mathfrak{G}(U)$ acts transitively on $U$, then the connected component of the unit element of $\mathfrak{G}$ also acts transitively on $U$. This follows from the fact that $U$ is connected. Also, we easily establish the following

**Proposition 11.** *Let the affine group $\mathfrak{G}$ act transitively in the convex domain $U$ of the affine space $P$. If the subspace $R_1 \subset R_P$ is invariant under the linear parts of all the transformations in $\mathfrak{G}$, then for any $x_0 \in U$ the cross-section of $U$ by the linear manifold $x_0 + R_1$ is a convex homogeneous domain in $x_0 + R_1$.*

**Proof.** Let $\mathfrak{G}_1$ be the subgroup of $\mathfrak{G}$ formed by the automorphisms $C \in \mathfrak{G}$ of the domain $U$ that leave the linear manifold $x_0 + R_1$ invariant. Since the subspace $R_1$ is invariant, it follows that every transformation $C \in \mathfrak{G}$ for which $Cx_0 \in x_0 + R_1$ belongs to $\mathfrak{G}_1$. It is now obvious that $\mathfrak{G}_1$ acts transitively on $U \cap (x_0 + R_1)$.

Every convex domain $U$ in the $n$-dimensional affine space $P$ can be put into correspondence with a convex cone $V(U)$ in the $(n + 1)$-dimensional linear space $R$. To do this we imbed the space $P$ in $R$ as a hyperplane not passing through 0 and we put

$$V(U) = \{\lambda x:\ x \in U,\ \lambda > 0\}. \tag{19}$$

Clearly $V(U)$ is a convex cone. Isomorphic convex domains correspond to isomorphic cones.

**Definition 9.** The convex cone $V(U)$ is said to be *the cone fitted onto the convex domain $U$.*

Conversely, every nonempty cross-section of a convex cone by a hyperplane not passing through 0 is a convex domain. However, different cross-sections may be nonisomorphic as convex domains.

If the domain $U$ is homogeneous, then so is the cone $V(U)$. For the automorphisms of $U$ can be continued by linearity into automorphisms of $V(U)$. If we then add the similarity transformations we obtain a group that is transitive on $V(U)$.

**Definition 10.** The *characteristic function* $\phi_U$ of a convex domain $U$ is the restriction to $U$ of the characteristic function of the cone $V(U)$.

With this definition Propositions 3, 4 and 5 remain valid for convex domains; the quadratic differential form $d^2 \ln \phi_U$ defines the *canonical Riemannian metric* on the convex domain $U$, which is invariant under all the automorphisms of $U$.

We shall give one example of a convex homogeneous domain which, in a certain sense, is typical. We consider the domain

$$x_0 > x_1^2 + \cdots + x_n^2 \tag{20}$$

(the interior of a paraboloid) in $(n+1)$-dimensional space. This domain is convex, contains no straight line and is invariant under the following two types of affine transformations:

1) $x_i \to x_i + a \quad (i = 1, \ldots, n); \quad x_0 \to x_0 + \sum_{i=1}^{n} (2a_i x_i + a_i^2)$;

2) $x_i \to \lambda x_i \ (i = 1, \ldots, n); \ x_0 \to \lambda^2 x_0$.

It is easy to see that the group generated by these transformations is transitive in the domain (20). The convex cone fitted onto this domain is isomorphic to the spherical cone in $(n+2)$-dimensional space (cf. the Introduction).

## §6. Stability and simply transitive groups of automorphisms

Proposition 12. *Let $U$ be a convex homogeneous domain in the affine space $P$ and let $\mathfrak{G}$ be a closed subgroup of the group $\mathfrak{G}(U)$ that acts transitively on $U$. The stability subgroup $\mathfrak{K} \subset \mathfrak{G}$ of any point $x_0 \in U$ is a maximal compact subgroup of the group $\mathfrak{G}$.*

Proof. Since the transformations in $\mathfrak{K}$ leave the point $x_0$ invariant, we may consider $\mathfrak{K}$ as a group of linear transformations of $R_P$ (cf. §5). Its compactness can be deduced either from the invariance of the positive-definite quadratic form $d^2 \ln \phi_U(x_0)$ or from the invariance of the bounded set $(U - x_0) \cap (x_0 - U) \subset R_P$.

Suppose now that $\mathfrak{K}_1$ is an arbitrary compact subgroup of $\mathfrak{G}$. We take a bounded open set $M \subset U$ and consider the set

$$\mathscr{K}_1 M = \{Cx : \ C \in \mathscr{K}_1, \quad x \in M\},$$

which is invariant under all the transformations in $\mathfrak{K}_1$. Since $\mathfrak{K}_1$ is compact, the set $\mathfrak{K}_1 M$ is bounded. Its center of gravity $x_1$ lies in $U$ and is a fixed point for $\mathfrak{K}_1$. (The center of gravity of a bounded open set in a real $n$-dimensional space is an affine invariant.) Thus, $\mathfrak{K}_1$ lies in the stability group of the point $x_1$ and therefore is conjugate to a subgroup $\mathfrak{K}_2$ of $\mathfrak{K}$. We now take $\mathfrak{K}_1$ to be a maximal compact subgroup of $\mathfrak{G}$. The subgroup $\mathfrak{K}_2$ is then also a maximal compact subgroup and coincides with $\mathfrak{K}$. This proves the proposition.

Corollary. *Under the conditions of Proposition 12, the number of connected components of the group $\mathfrak{G}$ is finite.*

Proof. Since $\mathfrak{G}$ acts transitively on $U$, so does its connected component of the unit element $\mathfrak{G}_0$. Therefore, every connected component of $\mathfrak{G}$ has a nonempty intersection with $\mathfrak{K}$ and the number of connected components of $\mathfrak{G}$ does not exceed the number of connected components of $\mathfrak{K}$. Our result now follows from the fact that the number of connected components of a compact group is finite.

We shall denote the connected component of the unit element of $\mathfrak{G}(U)$ by $\mathfrak{G}_0(U)$.

**Proposition 13.** *If $U$ is a convex homogeneous domain, then $\mathfrak{G}_0(U)$ coincides with the connected component of the unit element of its normalizer in the complete affine group.*

**Proof.** We denote the normalizer of $\mathfrak{G}_0(U)$ by $\mathfrak{N}$. Clearly, $\mathfrak{N} \supset \mathfrak{G}(U)$. Let $x_0 \in U$ and let the transformation $C \in \mathfrak{N}$ be so close to the identity that $Cx_0 \in U$. Then

$$CU = C\mathfrak{G}_0(U)x_0 = \mathfrak{G}_0(U)Cx_0 = U,$$

i.e., $C \in \mathfrak{G}(U)$. Thus, there exists a neighborhood of the identity in the complete affine group within which the groups $\mathfrak{N}$ and $\mathfrak{G}(U)$ coincide. This implies that these groups have identical connected components of the unit element, which is what we had to prove.

**Definition 11.** The group $\mathfrak{G}$ of affine transformations of an affine space $P$ is said to be *triangular* if the linear parts of the transformations in $\mathfrak{G}$ can be written as (upper) triangular matrices with respect to some basis.

When we imbed $P$ in a linear space of one more dimension (as in the construction of the cone $V(U)$ in §5), then $\mathfrak{G}$ is imbedded in a group of linear transformations of $R$. The group $\mathfrak{G}$ is triangular if and only if the corresponding group acting in $R$ is triangular.

It was proved in [4] that the maximal connected triangular subgroups of an arbitrary Lie group are intrinsically conjugate. By virtue of the remark made above, this theorem carries over to affine Lie groups.

It was also proved in [4] that if a linear group $\mathfrak{G}$ is the connected component of the unit element of an algebraic linear group, then it can be decomposed in the form

$$\mathfrak{G} = \mathfrak{K}\mathfrak{T}, \tag{21}$$

where $\mathfrak{K}$ is a connected compact subgroup and $\mathfrak{T}$ is a connected triangular subgroup of $\mathfrak{G}$. This theorem also carries over to affine groups, if we adopt

**Definition 12.** An affine group is said to be *algebraic* if it is selected from the complete affine group by polynomial equations connecting the coefficients of an affine transformation in an affine coordinate system.

It is easy to see that an affine group is algebraic if and only if the linear group corresponding to it in the space of one more dimension is algebraic. This allows us to carry the decomposition (21) over to affine groups.

We shall make a few additional remarks about the decomposition (21). We first prove that the intersection of any compact subgroup $\mathfrak{K}_1$ of $\mathfrak{G}$ with any of its triangular subgroups $\mathfrak{T}_1$ consists of the unit element only. The transformations in $\mathfrak{K}_1$ are semisimple; their eigenvalues are equal to 1 in modulus. On the other hand, the transformations in $\mathfrak{T}_1$ have positive eigenvalues. These three conditions can only be satisfied simultaneously by the identity transformation, which is what we had to prove. This implies that in the decomposition (21) the groups $\mathfrak{K}$ and $\mathfrak{T}$ intersect only in the unit element, that $\mathfrak{K}$ is a maximal compact subgroup, and that $\mathfrak{T}$ is a maximal connected triangular subgroup of $\mathfrak{G}$. Further, the group $\mathfrak{T}$ can be replaced by any subgroup conjugate to it. In fact, for every $g = kt \in \mathfrak{G}$ $(k \in \mathfrak{K},\ t \in \mathfrak{T})$ we have

$$\mathcal{G} = \mathcal{G}g^{-1} = \mathcal{K}\mathcal{T}g^{-1} = (\mathcal{K}k)(t\mathcal{T})\,g^{-1} = \mathcal{K}\,(g\mathcal{T}g^{-1}).$$

This means that for $\mathfrak{T}$ in (21) we can take any maximal connected triangular subgroup of $\mathfrak{G}$. In the same way we can show that $\mathfrak{K}$ can be taken to be any maximal compact subgroup of $\mathfrak{G}$.

The basis for the application of the results of [4] to the group of automorphisms of a convex homogeneous domain is

Proposition 14. *If $U$ is a convex homogeneous domain, then the group $\mathfrak{G}_0(U)$ coincides with the connected component of the unit element of an affine algebraic group.*

Proof. Let $\mathfrak{N}$ be the normalizer of $\mathfrak{G}_0(U)$ that occurs in Proposition 13. We shall show that $\mathfrak{N}$ is an algebraic group, which will imply Proposition 14. The affine transformations $C \in \mathfrak{N}$ are characterized by the fact that

$$C\mathfrak{G}_0(U)\,C^{-1} \subset \mathfrak{G}_0(U).$$

We denote by Ad the associated linear representation of the group of nondegenerate affine transformations in the Lie algebra of infinitesimal affine transformations. Then the transformations $C \in \mathfrak{N}$ can also be characterized by the fact that

$$(\operatorname{Ad} C)\,G(U) \subset G(U). \tag{22}$$

(We recall that $G(U)$ is the Lie algebra of $\mathfrak{G}(U)$.) It is easy to see that the representation Ad is rational (in the linear case $(\operatorname{Ad} C)\,D = CDC^{-1}$), and therefore in coordinate form the condition (22) can be written as a number of polynomial relations between the coefficients of $C$.

Propositions 12 and 14 and the results of [4] referred to above lead us to the following theorem.

Theorem 1. *Let $U$ be a convex homogeneous domain in an affine space, $\mathfrak{G}(U)$ the group of all its automorphisms, $\mathfrak{K}(U)$ the stability subgroup of some* *point* $x_0 \in U$, $\mathfrak{T}(U)$ *a maximal connected triangular subgroup of* $\mathfrak{G}(U)$ (*cf. Definition* 12). *Then*

$$\mathfrak{G}(U) = \mathfrak{K}(U)\,\mathfrak{T}(U),$$

*where* $\mathfrak{K}(U) \cap \mathfrak{T}(U) = \{e\}$. *The group* $\mathfrak{T}(U)$ *acts simply transitively in* $U$.

We observe that if $U = V$ is a convex cone, then $\mathfrak{T}(V)$ contains all the similarity transformations with positive coefficients. In fact, the group generated by these transformations and $\mathfrak{T}(V)$ is connected and triangular and therefore coincides with $\mathfrak{T}(V)$.

**Corollary.** *For every convex homogeneous cone* $V$ *there exists a convex homogeneous domain* $U$ *of dimension one less such that* $V$ *is isomorphic to the cone fitted onto* $U$ (*cf. Definition* 9).

**Proof.** By Theorem 1, there exists a triangular group $\mathfrak{T} \subset \mathfrak{G}(V)$ that acts transitively on $V$. Like every triangular linear group, it has an invariant subspace $R_1$ of codimension 1. Let $x_0$ be any point of $V$. It follows from Proposition 11 that the convex domain $U = V \cap (x_0 + R_1)$ is homogeneous. Since $\mathfrak{T}x_0 = V$, the subspace $R_1$ cannot contain $x_0$ and the hyperplane $x_0 + R_1$ does not pass through 0. Therefore $V \simeq V(U)$.

## CHAPTER II

## THE APPLICATION OF LEFT-SYMMETRIC ALGEBRAS

### §1. The algebra of a convex homogeneous domain

We shall use the notation introduced in §5 of Chapter I.

Suppose that in an affine space $P$ we are given a convex homogeneous domain $U$. By Theorem 1, there exists a triangular affine group $\mathfrak{T}(U)$ that acts simply transitively on $U$. We denote its Lie algebra by $T(U)$. If $x_0$ is any point of $U$, then the mapping

$$D \longrightarrow D(x_0) \quad (D \in T(U))$$

is an isomorphism of the linear space $T(U)$ onto $R_P$. Let $D_a$ be the inverse image of the vector $a \in R_P$ under this mapping, i.e.,

$$D_a(x_0) = a. \tag{1}$$

Let $L_a$ denote the linear part of the operator $D_a$. We now define an operation of multiplication $\Delta$ in $R_P$ by the formula

$$a \Delta b = L_a b. \tag{2}$$

**Definition 1.** The space $R_P$ with the operation of multiplication $\Delta$ defined by (2) is called the *algebra of the convex homogeneous domain* $U$ *with respect to the point* $x_0 \in U$ *and the transitive connected triangular group* $\mathfrak{T}(U)$.

Since $U$ is a homogeneous domain and transitive connected triangular groups are conjugate, it follows that a different choice of the point $x_0$ and the group $\mathfrak{T}(U)$ would lead us to an isomorphic algebra. Therefore we may simply speak of the *algebra of the convex homogeneous domain* $U$, regarding it as an abstract algebra. We shall denote this algebra by $\mathfrak{L}(U)$. Clearly, isomorphic domains correspond to isomorphic algebras.

It follows from (1) and (2) that for any $a \in R_P$, $x \in P$

$$D_a(x) = a \Delta (x - x_0) + a. \tag{3}$$

The domain $U$ is the orbit of the affine group $\mathfrak{T}(U)$ generated by the infinitesimal transformations $D_a$. Therefore, it is completely determined where the algebra $\mathfrak{L}(U)$ is specified.

In the particular case when the domain in question is a *convex homogeneous cone* $V$ in a linear space $R$, we shall *regard the structure of* $\mathfrak{L}(V)$ *with respect to any point* $x_0 \in V$ *and the group* $\mathfrak{T}(V)$ *to be defined within* $R$ *itself.* (The space of free vectors in $R$ is naturally identified with $R$.) The infinitesimal transformations $D_a$ are then linear and they can be identified with their linear parts $L_a$.

We shall now establish some properties of the algebra $\mathfrak{L}(U)$ of a convex homogeneous domain $U$. We shall use the following notation:

$$L_a b = a \triangle b, \quad R_a b = b \triangle a,$$

$$[a \triangle b] = a \triangle b - b \triangle a,$$

$$[a \triangle b \triangle c] = a \triangle (b \triangle c) - (a \triangle b) \triangle c.$$

By the commutation rule for infinitesimal affine transformations (formula (17) of Chapter I)

$$[D_a, D_b](x_0) = L_a b - L_b a = [a \triangle b],$$

whence

$$[D_a, D_b] = D_{[a \triangle b]},$$

and their linear parts satisfy

$$[L_a, L_b] = L_{[a \triangle b]}. \tag{4}$$

Definition 2. An algebra satisfying the condition (4) is said to be *left-symmetric.*

The condition (4) can also be written in the following equivalent forms:

$$[a \triangle b \triangle c] = [b \triangle a \triangle c], \tag{5}$$

$$[L_a, R_b] = R_{a \triangle b} - R_b R_a. \tag{6}$$

Equation (5) explains the term "left-symmetric algebra" (by analogy with left-alternative algebras).

Suppose further that $\phi = \phi_U$ is the characteristic function of $U$ (cf. §5 of Chapter I).

We shall calculate the first few terms in the Taylor series expansion of $\ln\phi$ in the neighborhood of $x_0 \in U$. We may assume that $\phi(x_0) = 1$. Further, for every $a \in R_P$

$$\varphi((\exp D_a)x_0) = (\det \exp L_a)^{-1}\varphi(x_0) = e^{-\mathrm{Sp}\, L_a},$$

since $\exp L_a$ is the linear part of the affine transformation $\exp D_a$. Taking logarithms we find that

$$\ln\varphi((\exp D_a)x_0) = -\mathrm{Sp}\, L_a.$$

By virtue of formula (18) of Chapter I

$$(\exp D_a)x_0 = x_0 + \sum_{k=0}^{\infty}\left(\frac{L_a^k}{(k+1)!}\right)a = x_0 + a + \frac{1}{2}a\triangle a + \frac{1}{6}a\triangle(a\triangle a) + \dots .$$

We put

$$s(a) = \mathrm{Sp}\, L_a.$$

Then

$$\ln\varphi\left(x_0 + a + \frac{1}{2}a\triangle a + \frac{1}{6}a\triangle(a\triangle a) + \dots\right)$$
$$= (d\ln\varphi(x_0))\left(a + \frac{1}{2}a\triangle a + \frac{1}{6}a\triangle(a\triangle a)\right)$$
$$+ \frac{1}{2}(d^2\ln\varphi(x_0))\left(a + \frac{1}{2}a\triangle a\right) + \frac{1}{6}(d^3\ln\varphi(x_0))(a) + \dots = -s(a).$$

If we denote by $g(x_0)$ the symmetric bilinear form connected with the quadratic form $d^2\ln\phi(x_0)$, then

$$(d^2\ln\varphi(x_0))\left(a + \frac{1}{2}a\triangle a\right) = (g(x_0))\left(a + \frac{1}{2}a\triangle a,\ a + \frac{1}{2}a\triangle a\right)$$
$$= (d^2\ln\varphi(x_0))(a) + \frac{1}{4}(d^2\ln\varphi(x_0))(a\triangle a) + (g(x_0))(a,\ a\triangle a).$$

If we omit the terms of higher than the third order of smallness with respect to $a$, we finally obtain

$$(d\ln\varphi(x_0))(a) + \frac{1}{2}[(d\ln\varphi(x_0))(a\triangle a) + (d^2\ln\varphi(x_0))(a)]$$
$$+ \frac{1}{6}[(d\ln\varphi(x_0))(a\triangle(a\triangle a)) + 3(g(x_0))(a,\ a\triangle a) + (d^3\ln\varphi(x_0))(a)] + \dots$$
$$= -s(a).$$

Comparing terms of the first order of smallness we find that

$$(d \ln \phi(x_0))(a) = -s(a). \tag{7}$$

Then, by comparing terms of the second order of smallness, we have

$$(d^2 \ln \phi(x_0))(a) = s(a \Delta a). \tag{8}$$

It follows from (4) that

$$s([a \Delta b]) = 0, \tag{9}$$

i.e., the bilinear form $s(a \Delta b)$ is symmetric. Therefore

$$g(x_0)(a, b) = s(a \Delta b). \tag{10}$$

Finally, from the quantities of the third order of smallness we find, using (7) and (10), that

$$(d^3 \ln \phi(x_0))(a) = -2s(a \Delta (a \Delta a)). \tag{11}$$

Since $\ln \phi$ is convex, (8) implies that

$$s(a \Delta a) > 0 \quad \text{if} \quad a \neq 0. \tag{12}$$

Definition 3. A left-symmetric algebra in which there exists a linear form $s$ satisfying the conditions (9) and (12) is said to be *compact*.

Thus, the algebra of $U$ is a compact left-symmetric algebra.

The group $\mathfrak{T}(U)$ is triangular; therefore the linear transformations $L_a$, $a \in R_p$, are simultaneously reducible to triangular form and have real eigenvalues.

Definition 4. A compact left-symmetric algebra in which the operators of left multiplication have only real eigenvalues is called a *clan*.

To summarize the results we have obtained, we may formulate

Proposition 1. *The algebra of any convex homogeneous domain is a clan.*

Let $U = V$ be a convex homogeneous cone in the linear space $R$. Then, for every $a \in R$

$$L_a x_0 = a \Delta x_0 = a,$$

i.e., $x_0$ is a right unit element in the algebra of $V$ with respect to $x_0$. It is not difficult to see the effect of the transformation $L_{x_0}$ in the space $R$. The group $\mathfrak{T}(V)$ contains the homothetic transformations with positive coefficients. Therefore its Lie algebra $T(V)$ contains the identity transformation $E$ of the space $R$. Since $Ex_0 = x_0$, it follows that $L_{x_0} = E$ and

$$x_0 \Delta x = x$$

for every $x \in R$. This means that $x_0$ is not only a right unit but also a left unit in the algebra of $V$ with respect to $x_0$. This proves

Proposition 2. *The algebra of any convex homogeneous cone has a unit element.*

Finally, we indicate the connection between the algebras of the domain $U$

and the cone $V(U)$.

We say that an algebra $\mathfrak{L}_1$ is obtained from $\mathfrak{L}$ by adjunction of a unit element if $\mathfrak{L}$ is imbedded in $\mathfrak{L}_1$ in such a way that

$$\mathfrak{L}_1 = \mathfrak{L} \dotplus \{\lambda e\},$$

where $e$ is the unit element of $\mathfrak{L}_1$.

Proposition 3. *The algebra of the cone $V(U)$ fitted onto the convex homogeneous domain $U$ is obtained from the algebra of $U$ by adjunction of a unit element.*

Proof. Suppose that the affine space $P \supset U$ is, as usual, imbedded in the linear space $R$ in the form of a hyperplane not passing through 0. The transformations in $\mathfrak{T}(U)$ can be continued canonically to automorphisms of the cone $V(U) \subset R$. The direct product of the group $\mathfrak{T}(U)$ and the one-parameter group of homothetic transformations of the space $R$ is a maximal connected triangular group of automorphisms of the cone $V(U)$. Its Lie algebra $T(V(U))$ splits into the direct sum

$$T(V(U)) = T(U) \dotplus \{\lambda E\}.$$

The space $R_P$ is naturally imbedded into the space $R$ in the form of a subspace of codimension 1. Let $x_0 \in U \subset V(U)$. If $a \in R_P$, then $D_a \in T(U)$ and

$$D_a(P) \subset R_P.$$

This shows that the structure of the algebra of $U$ with respect to $x_0$ is the restriction to $R_P$ of the structure of the algebra of $V(U)$ defined on $R$. Moreover,

$$R = R_P \dotplus \{\lambda x_0\}$$

and $x_0$ plays the part of the unit element in the algebra of $V(U)$ with respect to $x_0$.

## §2. The construction of a convex homogeneous domain from its clan

Let $\mathfrak{L}$ be a given clan. We choose a point $x_0 \in \mathfrak{L}$ and consider the infinitesimal affine transformations

$$D_a\colon\ x \longrightarrow a \triangle (x - x_0) + a = L_a(x - x_0) + a \quad (a,\ x \in \mathfrak{L}) \tag{13}$$

of $\mathfrak{L}$. By formula (17) of Chapter 1

$$[D_a,\ D_b](x) = [L_a,\ L_b](x - x_0) + L_a b - L_b a = L_{[a\triangle b]}(x - x_0) + [a \triangle b],$$

whence

$$[D_a,\ D_b] = D_{[a\triangle b]},$$

so that the transformations $D_a$ form a Lie algebra. We denote it by $T(\mathfrak{L})$ and the affine Lie group corresponding to it by $\mathfrak{T}(\mathfrak{L})$. The linear parts $L_a$ of the transformations $D_a$ also form a Lie algebra. By the definition of a clan the operators

$L_a$ have only real eigenvalues. This implies that they can be simultaneously reduced to triangular form ([4], Proposition 2) and this means that the group $\mathfrak{T}(\mathfrak{L})$ is triangular.

Let $U(\mathfrak{L})$ be the orbit of $\mathfrak{T}(\mathfrak{L})$ in $\mathfrak{L}$ passing through $x_0$. The mapping $C \to Cx_0$ of $\mathfrak{T}(\mathfrak{L})$ into $\mathfrak{L}$ is regular at the unit element of $\mathfrak{T}(\mathfrak{L})$, because its principal linear part

$$D_a \to D_a(x_0) = a$$

is an isomorphic mapping of $T(\mathfrak{L})$ onto $\mathfrak{L}$. Therefore, $x_0$ is an interior point of the set $U(\mathfrak{L}) \subset \mathfrak{L}$. Since the set $U(\mathfrak{L})$ is homogeneous under the affine group $\mathfrak{T}(\mathfrak{L})$, all its points are interior.

Thus, $\mathfrak{T}(\mathfrak{L})$ acts transitively in the open set $U(\mathfrak{L})$ of $\mathfrak{L}$ containing $x_0$.

If in the above construction we use the point $\widetilde{x}_0$ instead of $x_0$, then the domain $U(\mathfrak{L})$ is merely displaced by $\widetilde{x}_0 - x_0$. For under the displacement of $\mathfrak{L}$ by $\widetilde{x}_0 - x_0$ the point $x$ goes over into $\widetilde{x}_0$ and the infinitesimal transformations $D_a$ into the transformations

$$\widetilde{D}_a\colon\ x + \widetilde{x}_0 - x_0 \to a \triangle (x - x_0) + a$$

or, if we make the change of variables $x + \widetilde{x}_0 - x_0 = y$,

$$\widetilde{D}_a\colon\ y \to a \triangle (y - \widetilde{x}_0) + a.$$

For $x_0$ we can take 0, for example. However, this is not always convenient. If the algebra $\mathfrak{L}$ has a unit element $e$, it is more convenient to put $x_0 = e$; then

$$D_a\ (x) = a \triangle (x - e) + a = a \triangle x,$$

i.e., $D_a = L_a$. We shall show that in this case *the domain* $U(\mathfrak{L}) = V(\mathfrak{L})$ *is a convex cone in* $\mathfrak{L}$. First, we observe that for every number $\lambda$ and for every $x \in V(\mathfrak{L})$

$$(\exp L_{\lambda e})\, x = (\exp \lambda E)\, x = e^{\lambda} x \in V\,(\mathfrak{L}).$$

Further, let $s$ be a linear form on $\mathfrak{L}$, satisfying the conditions (9) and (12). The subspace

$$T_0 = \{L_a : s(a) = 0\}$$

of the Lie algebra $T(\mathfrak{L})$ contains its derived algebra, by (4) and (9), and is therefore an ideal. It follows from (12) that

$$s(e) = s(e \triangle e) > 0.$$

Therefore $T(\mathfrak{L})$ splits into a direct sum of ideals:

$$T(\mathfrak{L}) = T_0 + \{\lambda e\}.$$

The group $\mathfrak{T}(\mathfrak{L})$ corresponding to it is simply connected (cf., for example, [4]) and splits into a direct product of the normal subgroup $\mathcal{T}_0$ corresponding to $T_0$, and the one-dimensional normal subgroup $\{\lambda E\}_{\lambda>0}$:

$$\mathcal{T}(\mathfrak{L}) = \mathcal{T}_0 \times \{\lambda E\}_{\lambda>0}.$$

We denote the orbit of $\mathcal{T}_0$ passing through $e \in V(\mathfrak{L})$ by $S$. Clearly

$$V(\mathfrak{L}) = \bigcup_{\lambda>0} \lambda S. \tag{14}$$

The tangent hyperplane to $S$ at $e$ is

$$P = \{e + a:\ s(a) = 0\} = \{x:\ s(x) = s(e)\}.$$

For every $a \in T_0$ we have

$$(\exp L_a)\, e = e + a + \frac{1}{2} a \triangle a + \dots .$$

It follows from (12) that $S$ lies entirely to one side of $P$ in some neighborhood of $e$ (the same side as the point $2e$). In other words, $S$ is convex at $e$. Since $S$ is homogeneous under the linear group $\mathcal{T}_0$, this implies that it is convex at all points. Therefore, for every $x \in S$

$$s(x) \geq s(e)$$

and, for any $\lambda > 0$,

$$s(\lambda x) \geq \lambda s(e) > 0.$$

Therefore

$$s(x) > 0$$

for all $x \in V(\mathfrak{L})$. If $C \in \mathcal{T}(\mathfrak{L})$, then

$$(C' s)(x) = s(Cx) > 0$$

for all $x \in V(\mathfrak{L})$. The mapping $C \to C's$ of $\mathcal{T}(\mathfrak{L})$ into the space $\mathfrak{L}'$ adjoint to $\mathfrak{L}$ is regular at the unit element of $\mathcal{T}(\mathfrak{L})$. For its principal linear part

$$L_a \to L'_a s \quad (a \in \mathfrak{L})$$

is an isomorphism of $T(\mathfrak{L})$ onto $\mathfrak{L}'$, because for any $a \in \mathfrak{L}$

$$(L'_a s)(a) = s(a \,\Delta\, a) > 0$$

and $L'_a \neq 0$. Therefore, the linear forms of the type $C's$, $C \in \mathcal{T}(\mathfrak{L})$, fill a neighborhood of the form $s$. All these forms are positive on $V(\mathfrak{L})$. Therefore the convex hull $W$ of $V(\mathfrak{L})$ contains no straight line. Clearly, $W$ is invariant under $\mathcal{T}(\mathfrak{L})$. We consider the canonical Riemannian metric in the convex cone $W$ (cf. §3 of Chapter I). It is invariant under $\mathcal{T}(\mathfrak{L})$. The set $V(\mathfrak{L}) \subset W$ is homogeneous and so is complete in this metric and therefore coincides with $W$. Thus, $V(\mathfrak{L})$ is a convex cone in $\mathfrak{L}$. Incidentally we have proved that

$$s \in (V(\mathfrak{L}))'. \tag{15}$$

Now let $\mathfrak{L}$ be an arbitrary clan, not necessarily containing a unit element. We shall show that $U(\mathfrak{L})$ is a convex domain in $\mathfrak{L}$. We consider the algebra $\mathfrak{L}_1$, obtained from $\mathfrak{L}$ by adjunction of a unit element:

$$\mathfrak{L}_1 = \mathfrak{L} \dotplus \{\lambda e\}.$$

It is easy to see that $\mathfrak{L}_1$ is left-symmetric and that the operators of left multiplication in it have only real eigenvalues. Suppose further that $s$ is a linear form on $\mathfrak{L}$ satisfying (9) and (12). Let $\alpha$ be the maximum value of $s$ on the unit sphere $s(a \Delta a) = 1$. Then, for all $a \in \mathfrak{L}$

$$(s(a))^2 \leq \alpha^2 s(a \Delta a).$$

We now extend the form $s$ to $\mathfrak{L}_1$ by putting

$$s(e) = 1 + \alpha^2,$$

and we verify that the so extended linear form $s$ still satisfies the conditions (9) and (12). For any $a, b \in \mathfrak{L}$ we have

$$s([(\lambda e + a) \triangle (\mu e + b)]) = s([a \triangle b]) = 0.$$

Further

$$s((\lambda e + a) \triangle (\lambda e + a)) = \lambda^2 s(e) + 2\lambda s(a) + s(a \triangle a).$$

Since, for $a \neq 0$

$$s(e)\, s(a \triangle a) - (s(a))^2 = (1 + \alpha^2)\, s(a \triangle a) - (s(a))^2 > 0,$$

it follows that, for $\lambda e + a \neq 0$,

$$s((\lambda e + a) \triangle (\lambda e + a)) > 0.$$

This proves that the algebra $\mathfrak{L}_1$ is a clan.

We consider the mapping

$$x \longrightarrow e + x$$

of $\mathfrak{L}$ onto the hyperplane $e + \mathfrak{L} \subset \mathfrak{L}_1$. The infinitesimal affine transformations

$$D_a : x \longrightarrow a \Delta x + a \quad (a \in \mathfrak{L})$$

of $\mathfrak{L}$ then go over into

$$e + x \longrightarrow a \Delta x + a = a \Delta (e + x),$$

i.e., into bounded infinitesimal transformations $L_a$, $a \in \mathfrak{L}$, of $\mathfrak{L}_1$ onto $e + \mathfrak{L}$. Therefore the set $U(\mathfrak{L})$ (constructed for $x_0 = 0$) goes over, under the above mapping, into the orbit of the linear group $\mathfrak{T}$, generated by the infinitesimal transformations $L_a$, $a \in \mathfrak{L}$, of $\mathfrak{L}_1$. Since $[\mathfrak{L} \Delta \mathfrak{L}_1] \subset \mathfrak{L}$, the infinitesimal transformations $L_a$, $a \in \mathfrak{L}$, form an ideal in the Lie algebra $T(\mathfrak{L}_1)$ and the linear group $\mathfrak{T}$ generated by them is a normal subgroup of $\mathfrak{T}(\mathfrak{L}_1)$. Therefore

$$\mathfrak{T}(\mathfrak{L}_1) = \mathfrak{T} \times \{\lambda E\}_{\lambda > 0},$$

and the orbit of $\mathfrak{T}$ passing through $e$ coincides with the intersection of the hyperplane $e + \mathfrak{L}$ with the orbit of $\mathfrak{T}(\mathfrak{L}_1)$, i.e., with the cone $V(\mathfrak{L}_1)$. Thus,

$$e + U(\mathfrak{L}) = (e + \mathfrak{L}) \cap V(\mathfrak{L}_1).$$

By what we have shown above, $V(\mathfrak{L}_1)$ is a convex cone in $\mathfrak{L}_1$. Therefore $U(\mathfrak{L})$ is a convex domain in $\mathfrak{L}$.

Comparison of the formulae (3) and (13) shows that the algebra of $U(\mathfrak{L})$ with respect to $x_0$ and $\mathfrak{T}(\mathfrak{L})$ coincides with $\mathfrak{L}$. This proves

**Theorem 2.** *The mapping that assigns to every convex homogeneous domain its algebra (cf. §1) is a one-to-one mapping of the set of convex homogeneous domains onto the set of clans. Moreover, the convex homogeneous cones, and only these, correspond to clans with a unit element.*

Of course it is assumed in the theorem that the convex homogeneous domains, and the clans, are considered to within an isomorphism.

The definition of a clan immediately implies that every subalgebra $\mathfrak{L}_1$ of a clan $\mathfrak{L}$ is a clan. Every such subalgebra is connected with a plane cross-section of $U(\mathfrak{L})$ that is homogeneous under a subgroup of $\mathfrak{T}(\mathfrak{L})$.

**Proposition 4.** *Let $\mathfrak{L}_1$ be a subalgebra of the clan $\mathfrak{L}$. The convex domain*

$$(x_0 + \mathfrak{L}_1) \cap U(\mathfrak{L})$$

*of the affine space $x_0 + \mathfrak{L}_1$ is homogeneous under a subgroup of $\mathfrak{T}(\mathfrak{L})$ and is isomorphic to $U(\mathfrak{L}_1)$.*

Proof. The transformations $D_a$, $a \in \mathfrak{L}_1$, form a subalgebra of $T(\mathfrak{L})$, which we denote by $T_1(\mathfrak{L})$. For any $a$, $x \in \mathfrak{L}_1$, we have

$$D_a(x_0 + x) = a \Delta x + a \in \mathfrak{L}_1. \tag{16}$$

This implies that the subgroup $\mathfrak{T}_1(\mathfrak{L})$ of $\mathfrak{T}(\mathfrak{L})$ corresponding to $T_1(\mathfrak{L})$ of $T(\mathfrak{L})$ leaves the linear manifold $x_0 + \mathfrak{L}_1$ invariant. Since $\mathfrak{T}(\mathfrak{L})$ acts simply transitively on $U(\mathfrak{L})$, for every point $x \in U(\mathfrak{L})$ the mapping

$$D_a \longrightarrow D_a(x) \quad (a \in \mathfrak{L})$$

is an isomorphism of $T(\mathfrak{L})$ onto $\mathfrak{L}$. Consequently, for $x \in U(\mathfrak{L}) \cap (x_0 + \mathfrak{L})$ the restriction of this mapping to $T_1(\mathfrak{L})$ is an isomorphism of $T_1(\mathfrak{L})$ onto a subspace $\mathfrak{L}_2 \subset \mathfrak{L}_1$. Dimensional arguments show that $\mathfrak{L}_2 = \mathfrak{L}_1$. This shows that $\mathfrak{T}_1(\mathfrak{L})$ acts transitively in the domain $(x_0 + \mathfrak{L}_1) \cap U(\mathfrak{L})$ of the affine space $x_0 + \mathfrak{L}_1$.

The formula (16) shows that $\mathfrak{T}_1(\mathfrak{L})$ acts in $x_0 + \mathfrak{L}_1$ in the same way as $\mathfrak{T}(\mathfrak{L}_1)$ in $\mathfrak{L}_1$. Therefore the domains $(x_0 + \mathfrak{L}_1) \cap U(\mathfrak{L})$ and $U(\mathfrak{L}_1)$ are isomorphic.

## §3. Principal decomposition

Let $\mathfrak{L}$ be a clan, $s$ a linear form on $\mathfrak{L}$ satisfying conditions (9) and (12). The scalar product

$$(a, b) = s(a \Delta b) \tag{17}$$

introduces into $\mathfrak{L}$ the structure of a Euclidean space. There exists a unique element $u \in \mathfrak{L}$ such that

$$s(a) = (u, a) \tag{18}$$

for all $a \in \mathfrak{L}$. We shall show that the operator $R_u$ of right multiplication by $u$ in $\mathfrak{L}$ is symmetric in the metric (17). For any $a, b \in \mathfrak{L}$ we have

$$(a, R_u b) - (b, R_u a) = s(a \triangle (b \triangle u) - b \triangle (a \triangle u)) = s([L_a, L_b]u)$$
$$= s(L_{[a \triangle b]}u) = ([a \triangle b], u) = s([a \triangle b]) = 0.$$

Further, for every $a \in \mathfrak{L}$

$$(u \triangle u, a) = (R_u u, a) = (u, R_u a) = s(R_u a) = (u, a),$$

which implies that $u \Delta u = u$, i.e., that $u$ is an idempotent of $\mathfrak{L}$.

Definition 5. The element $u$ defined by (18) is called the *principal idempotent* of the clan $\mathfrak{L}$.

If $\mathfrak{L}$ has a unit element $e$, then $u = e$, because

$$(e, a) = s(e \Delta a) = s(a).$$

The identity (16) implies that

$$[L_u, R_u] = R_u - R_u^2.$$

The operator $P = R_u - R_u^2$ is symmetric and commutes with $R_u$. Further,

$$\operatorname{Sp} P^2 = \operatorname{Sp} [L_u, R_u] P = \operatorname{Sp} L_u [R_u, P] = 0.$$

Therefore $P = 0$ and so

$$R_u^2 = R_u, \tag{19}$$

$$[L_u, R_u] = 0. \tag{20}$$

The equation (19) means that $R_u$ is an orthogonal projector onto a subspace $\mathfrak{L}_0 \subset \mathfrak{L}$. Thus, $\mathfrak{L}$ can be split into the sum of orthogonal subspaces,

$$\mathfrak{L} = \mathfrak{L}_0 + \mathfrak{N}, \tag{21}$$

in such a way that the operator $R_u$ is the identity on $\mathfrak{L}_0$ and annihilates all the vectors in $\mathfrak{N}$. The decomposition (21) is called the *principal decomposition of the clan* $\mathfrak{L}$. Clearly, if the idempotent $u$ lies in $\mathfrak{L}_0$, so $\mathfrak{L}_0 \neq 0$. In a clan with a unit element $\mathfrak{L}_0 = \mathfrak{L}$.

We now find the operator $L_u$. For all $a, b \in \mathfrak{L}$ we have

$$(L_u a,\ b) + (a,\ L_u b) = s\,(L_{u\triangle a} b + L_a L_u b)$$
$$= s\,(L_{a\triangle u} b + L_u L_a b) = (R_u a,\ b) + (a,\ b).$$

If we denote by $L_u^*$, the operator adjoint to $L_u$ with respect to the metric (17), then

$$L_u + L_u^* = R_u + E.$$

This means that the operator

$$K = \frac{1}{2}(R_u + E) - L_u$$

is skew-symmetric. It follows from (20) and the fact that the eigenvalues of $L_u$ are real that the eigenvalues of $K$ are also real. This can only occur if $K = 0$. Thus,

$$L_u = \frac{1}{2}(R_u + E), \tag{22}$$

i.e., the operator $L_u$ is the identity on $\mathfrak{L}_0$ and coincides with $\frac{1}{2}E$ on $\mathfrak{N}$.

**Lemma 1.** *If in a left-symmetric algebra*

$$L_c a = \lambda_a a, \quad L_c b = \lambda_b b, \quad R_c a = \mu a,$$

*for elements $a, b, c$, then*

$$L_c(a \triangle b) = (\lambda_a + \lambda_b - \mu)\, a \triangle b.$$

**Proof.** Using the identity (5) we find that

$$L_c(a \triangle b) = c \triangle (a \triangle b) = (c \triangle a) \triangle b + [c \triangle a \triangle b] = \lambda_a a \triangle b + [a \triangle c \triangle b]$$
$$= \lambda_a a \triangle b + a \triangle (c \triangle b) - (a \triangle c) \triangle b = (\lambda_a + \lambda_b - \mu)\, a \triangle b,$$

which is what we had to prove.

We shall apply Lemma 1 to the case where $c = u$, the principal idempotent of the clan $\mathfrak{L}$, and the elements $a$ and $b$ are taken from the subspaces $\mathfrak{L}_0$ and $\mathfrak{N}$ in different combinations. Then we obtain the following "multiplication table" in $\mathfrak{L}$:

| | $\mathfrak{L}_0$ | $\mathfrak{N}$ |
|---|---|---|
| $\mathfrak{L}_0$ | $\mathfrak{L}_0$ | $\mathfrak{N}$ |
| $\mathfrak{N}$ | 0 | $\mathfrak{L}_0$ |

(23)

From this table it is clear, in particular, that $\mathfrak{L}_0$ is a subalgebra. The idempotent $u \in \mathfrak{L}_0$ serves as a unit element for $\mathfrak{L}_0$. If $a, b \in \mathfrak{N}$, then

$$R_u[a \triangle b] = L_{[a\triangle b]} u = [L_a,\ L_b]\, u = L_a R_u b - L_b R_u a = 0,$$

i.e., $[a \Delta b] \in \mathfrak{N}$. On the other hand, (23) implies that $[a \Delta b] \in \mathfrak{L}_0$. This means that $[a \Delta b] = 0$. Thus, in addition to (23), we have

$$[\mathfrak{N} \triangle \mathfrak{N}] = 0. \tag{24}$$

Finally, for all $a \in \mathfrak{L}_0$, $x \in \mathfrak{N}$

$$[L_a, L_x] = L_{a\triangle x},$$

since $x \Delta a = 0$. Applying both sides of this equation to the vector $y \in \mathfrak{N}$ we find that

$$L_a(x \triangle y) = L_a x \triangle y + x \triangle L_a y.$$

Therefore

$$(\exp L_a)(x \triangle y) = (\exp L_a) x \triangle (\exp L_a) y \tag{25}$$

for arbitrary $a \in \mathfrak{L}_0$, $x, y \in \mathfrak{N}$.

Proposition 5. *Let $\mathfrak{L} = \mathfrak{L}_0 + \mathfrak{N}$ be the principal decomposition of the clan $\mathfrak{L}$, and let $u$ be its principal idempotent. If $U(\mathfrak{L})$ and $U(\mathfrak{L}_0)$ are the convex domains in $\mathfrak{L}$ and $\mathfrak{L}_0$, respectively, constructed according to the rules of §2 with $x_0 = u$, then $U(\mathfrak{L}_0) = V(\mathfrak{L}_0)$ is a cone and*

$$U(\mathfrak{L}) = \left\{x + y\colon\ x \in \mathfrak{L}_0,\ y \in \mathfrak{N},\ x - \frac{1}{2} y \triangle y \in V(\mathfrak{L}_0)\right\}$$

Proof. As we remarked above, the idempotent $u$ is the unit element of $\mathfrak{L}_0$. This implies that $V(\mathfrak{L}_0)$ is a convex cone in $\mathfrak{L}_0$. The group $\mathfrak{T}(\mathfrak{L}_0)$ acts transitively on $V(\mathfrak{L}_0)$ and is generated by the infinitesimal transformations

$$x \longrightarrow a \Delta x \quad (a, x \in \mathfrak{L}_0).$$

The group $\mathfrak{T}(\mathfrak{L})$ is transitive on $U(\mathfrak{L})$ and is generated by the infinitesimal transformations

$$D_a\colon\ x + y \longrightarrow a \triangle (x + y - u) + a = a \triangle x + a \triangle y \ \ (a,\ x \in \mathfrak{L}_0,\ y \in \mathfrak{N})$$

and

$$D_b\colon\ x + y \longrightarrow b \triangle (x + y - u) + b = b \triangle y + b \ \ (x \in \mathfrak{L}_0,\ b,\ y \in \mathfrak{N}).$$

Using the same notation, we have

$$(\exp D_a)(x+y) = (\exp L_a) x + (\exp L_a) y, \tag{26}$$

$$(\exp D_b)(x + y) = x + y + b \triangle y + b + \frac{1}{2} b \triangle b$$

(cf. formula (18) of Chapter I). We rewrite the last relation, combining the terms in $\mathfrak{L}_0$ on one side and those in $\mathfrak{N}$ on the other:

$$(\exp D_b)(x + y) = \left(x + b \triangle y + \frac{1}{2} b \triangle b\right) + (y + b). \tag{27}$$

We shall show that the affine transformations (26) and (27), and therefore any transformations in $\mathfrak{T}(\mathfrak{L})$, leave the domain

$$\widetilde{U}(\mathfrak{L}) = \left\{x+y\colon\ x \in \mathfrak{L}_0,\ y \in \mathfrak{N},\ x - \frac{1}{2} y \triangle y \in V(\mathfrak{L}_0)\right\}$$

invariant. Let $x \in \mathfrak{L}_0$, $y \in \mathfrak{N}$ be such that $x + y \in \widetilde{U}(\mathfrak{L})$. By virtue of (25), for every $a \in \mathfrak{L}_0$

$$(\exp L_a)\, x - \frac{1}{2}(\exp L_a)\, y \triangle (\exp L_a)\, y = (\exp L_a)\left(x - \frac{1}{2} y \triangle y\right) \in V(\mathfrak{L}_0),$$

since $V(\mathfrak{L}_0)$ is invariant under the transformation $\exp L_a$. Further, using (24), we find that for every $b \in \mathfrak{N}$

$$x + b \triangle y + \frac{1}{2} b \triangle b - \frac{1}{2}(y+b) \triangle (y+b) = x - \frac{1}{2} y \triangle y \in V(\mathfrak{L}_0).$$

We shall now show that the group $\mathfrak{T}(\mathfrak{L}_0)$ acts transitively on $\widetilde{U}(\mathfrak{L})$. Let $x \in \mathfrak{L}_0$, $y \in \mathfrak{N}$, $x + y \in \widetilde{U}(\mathfrak{L})$. Then

$$(\exp D_{-y})(x+y) = x - \frac{1}{2} y \triangle y \in V(\mathfrak{L}_0) \subset U(\mathfrak{L}).$$

Further, there exists a transformation in $\mathfrak{T}(\mathfrak{L}_0)$ that transforms the point $x - \frac{1}{2}y \triangle y$ of $V(\mathfrak{L}_0)$ into the point $u \in V(\mathfrak{L}_0)$. It is clear from (26) that this transformation is the restriction to $\mathfrak{L}_0$ of a transformation $C \in \mathfrak{T}(\mathfrak{L})$. We have

$$(C \exp D_{-y})(x+y) = C\left(x - \frac{1}{2} y \triangle y\right) = u.$$

Thus, any point of $\widetilde{U}(\mathfrak{L})$ goes over into $u \in \widetilde{U}(\mathfrak{L})$ under some transformation of $\mathfrak{T}(\mathfrak{L})$. This implies that $\mathfrak{T}(\mathfrak{L})$ acts transitively on $\widetilde{U}(\mathfrak{L})$.

Thus, the sets $U(\mathfrak{L})$ and $\widetilde{U}(\mathfrak{L})$ are both orbits of $\mathfrak{T}(\mathfrak{L})$. Since they have the point $u$ in common, they must coincide.

Proposition 6. *In the notation of Proposition 5, for every $c \in U(\mathfrak{L})$ the cone $c + \overline{V(\mathfrak{L}_0)}$ is the largest cone with vertex at $c$ contained in $U(\mathfrak{L})$.*

Proof. Let $c = a + b$, where $a \in \mathfrak{L}_0$, $b \in \mathfrak{N}$. Then $a - \frac{1}{2} b \triangle b \in V(\mathfrak{L}_0)$. For every $x \in \overline{V(\mathfrak{L}_0)}$

$$a + x - \frac{1}{2} b \triangle b \in V(\mathfrak{L}_0),$$

so that $c + x \in U(\mathfrak{L})$.

Let $x \in \mathfrak{L}_0$, $y \in \mathfrak{N}$ be such that $c + \lambda(x+y) \in U(\mathfrak{L})$ for all $\lambda > 0$. This means that for $\lambda > 0$

$$a + \lambda x - \frac{1}{2}(b + \lambda y) \triangle (b + \lambda y) \in V(\mathfrak{L}_0)$$

or, if we divide by $\lambda^2$,

$$\frac{1}{\lambda^2} a + \frac{1}{\lambda} x - \frac{1}{2}\left(\frac{1}{\lambda} b + y\right) \triangle \left(\frac{1}{\lambda} b + y\right) \in V(\mathfrak{L}_0).$$

Passing to the limit, as $\lambda \to \infty$, we find that

$$-y \triangle y \in \overline{V(\mathfrak{L}_0)}.$$

If $s$ is a linear form on $\mathfrak{L}$ that satisfies the conditions (9) and (12), then its restriction to $\mathfrak{L}_0$ also has these properties. It follows from (15) that $s(y \Delta y) \leq 0$. This is only possible if $y = 0$; but then

$$x+y=x=\lim_{\lambda\to\infty}\frac{1}{\lambda}\left(a+\lambda x-\frac{1}{2}b \triangle b\right)\in\overline{V(\mathfrak{L}_0)},$$

which proves the proposition.

There was an arbitrary element in the construction of the subalgebra $\mathfrak{L}_0 \subset \mathfrak{L}$ connected with the choice of the form $s$. Proposition 6 shows that the subalgebra $\mathfrak{L}_0$ and, therefore, the principal idempotent $u$, which is its unit element, do not, in fact, depend on the choice of the form $s$. For the subspace $\mathfrak{L}_0 \subset \mathfrak{L}$ coincides with the linear hull of the cone $V(\mathfrak{L}_0)$, and by Proposition 6, the latter has an invariant geometrical significance.

Proposition 7. *In the notation of Proposition* 5, *for every* $y \in \mathfrak{R}$

$$y \triangle y \in \overline{V(\mathfrak{L}_0)}.$$

Proof. It follows from Proposition 5 that

$$\frac{1}{2}y \triangle y \pm y \in \overline{U(\mathfrak{L})}.$$

Since the set $\overline{U(\mathfrak{L})}$ is convex,

$$\frac{1}{2}y \triangle y=\frac{1}{2}\left[\left(\frac{1}{2}y \triangle y+y\right)+\left(\frac{1}{2}y \triangle y-y\right)\right]\in\overline{U(\mathfrak{L})},$$

but $\frac{1}{2}y \Delta y \in \mathfrak{L}_0$ and therefore $\frac{1}{2}y \Delta y \in \overline{V(\mathfrak{L}_0)}$.

## §4. Normal decomposition

In the present section we shall study clans with a unit element. The results we shall obtain are easily generalized to arbitrary clans, but we shall not need this in what follows.

Every nonzero element $v \in \mathfrak{L}$ for which $v \Delta v = v$ is called an *idempotent.*

Definition 6. *A normal decomposition of a clan* $\mathfrak{L}$ *with a unit element* is a decomposition of the space $\mathfrak{L}$ into the direct sum of subspaces

$$\mathfrak{L}=\sum_{i\leq j}\mathfrak{L}_{ij} \qquad (i,\ j=1,\ \ldots,\ m),$$

having the following properties:

1) for every $i$, $\mathfrak{L}_{ii}$ is one-dimensional and is generated by idempotent $e_i$;

2) the operators $L_i = L_{e_i}$ and $R_i = R_{e_i}$ leave the subspaces $\mathfrak{L}_{jk}$ invariant

and on each of them they reduce to multiplication by a number given by the following table:

| | $\mathfrak{L}_{ii}$ | $\mathfrak{L}_{ji}\,(j\neq i)$ | $\mathfrak{L}_{ik}\,(k\neq i)$ | $\mathfrak{L}_{jk}\,(j,\,k\neq i)$ |
|---|---|---|---|---|
| $L_i$ | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 |
| $R_i$ | 1 | 1 | 0 | 0 |

(28)

Obviously, in the notation of Definition 6, the element $\Sigma\, e_i$ is a unit element of $\mathfrak{L}$. Suppose further that $s$ is a linear form on $\mathfrak{L}$ satisfying the conditions (9) and (12). It follows from the equation $x = 2[e_i \,\Delta\, x]$ for $x \in \mathfrak{L}_{ij}$ $(i < j)$ that

$$s(\mathfrak{L}_{ij}) = 0 \qquad \text{for } i < j. \tag{29}$$

The main purpose of this section is to prove the existence and uniqueness of the normal decomposition.

**Lemma 2.** *Let $\mathfrak{L}$ be a clan with unit element $e$. There exist a subalgebra $\mathfrak{L}_1 \subset \mathfrak{L}$ and an idempotent $v \in \mathfrak{L}$ such that*

$$\mathfrak{L} = \mathfrak{L}_1 + \{\lambda v\}$$

*(direct sum of subspaces) and the element $u = e - v$ is the principal idempotent of the clan $\mathfrak{L}_1$.*

**Proof.** By the Corollary to Theorem 1, $V(\mathfrak{L})$ is isomorphic to the cone fitted onto a convex homogeneous domain $U$. If $\mathfrak{L}_1$ is the algebra of $U$, then $\mathfrak{L}$ is obtained from $\mathfrak{L}_1$ by adjunction of a unit element (Proposition 3). Let $u$ be the principal idempotent of the clan $\mathfrak{L}_1$. We put $v = e - u$. It is easy to see that $\mathfrak{L}_1$ and $v$ satisfy the requirements of the lemma.

**Proposition 8.** *Every clan with a unit element has a normal decomposition.*

We conduct the proof by induction on the dimension of the clan. Let $\mathfrak{L}$ be a clan with a unit element $e$. We apply Lemma 2 to it and form the principal decomposition of the clan $\mathfrak{L}_1$:

$$\mathfrak{L}_1 = \mathfrak{L}_0 + \mathfrak{N}$$

(cf. §3). The clan $\mathfrak{L}_0$ has the unit element $u$; by the induction hypothesis it has the normal decomposition

$$\mathfrak{L}_0 = \sum_{i\leq j} \mathfrak{L}_{ij} \qquad (i,\, j = 1,\, \ldots,\, m-1).$$

Let $e_i$ $(i = 1, \cdots, m-1)$ be the idempotent generating the subspace $\mathfrak{L}_{ii}$ and let $L_i = L_{e_i}$. It follows from the properties of the principal decomposition that the operators $L_i$ leave $\mathfrak{N}$ invariant and that for any $x, y \in \mathfrak{N}$

$$L_i(x \triangle y) = L_i x \triangle y + x \triangle L_i y. \tag{30}$$

Moreover, the operators $L_i$ commute with one another and have real eigenvalues. The space $\mathfrak{N}$ splits with respect to the operators $L_i$ into the direct sum of weight subspaces:

$$\mathfrak{N} = \Sigma \mathfrak{N}_\alpha,$$

where $\alpha = (\alpha_1, \cdots, \alpha_{m-1})$ are distinct collections of real numbers and the operators $L_i - \alpha_i E$ $(i = 1, \cdots, m-1)$ are nilpotent on the subspaces $\mathfrak{N}_\alpha$.

We examine the subspace $\mathfrak{N}_\alpha \Delta \mathfrak{N}_\alpha \subset \mathfrak{L}_0$. It follows from the definition of $\mathfrak{N}_\alpha$ and the relation (30) that the operators $L_i - 2\alpha_i E$ are nilpotent on $\mathfrak{N}_\alpha \Delta \mathfrak{N}_\alpha$. Since $s(\mathfrak{N}_\alpha \Delta \mathfrak{N}_\alpha) \neq 0$, there exists a $k$ such that the projection of $\mathfrak{N}_\alpha \Delta \mathfrak{N}_\alpha$ onto $\mathfrak{L}_{kk}$ is not zero (cf. (29)); but then $\alpha_i = \frac{1}{2}\delta_{ik}$. Moreover, this implies that the projection of $\mathfrak{N}_\alpha \Delta \mathfrak{N}_\alpha$ onto subspaces $\mathfrak{L}_{ij}$ different from $\mathfrak{L}_{kk}$ is zero, because on these subspaces the operators $L_i$ have different eigenvalues (cf. table (28)). Thus, $\alpha$ can only take the values $\alpha^{(1)}, \cdots, \alpha^{(m-1)}$, where $\alpha_i^{(k)} = \frac{1}{2}\delta_{ik}$; moreover $\mathfrak{N}_{\alpha(k)} \Delta \mathfrak{N}_{\alpha(k)} \subset \mathfrak{L}_{kk}$.

For $j < k$ the operators $L_j - \frac{1}{2}E$, $L_k - \frac{1}{2}E$, $L_i$ $(i \neq j, k)$ are nilpotent on the subspace $\mathfrak{N}_{\alpha(j)} \Delta \mathfrak{N}_{\alpha(k)} \subset \mathfrak{L}_0$; it is clear from table (28) that $\mathfrak{N}_{\alpha(j)} \Delta \mathfrak{N}_{\alpha(k)} \subset \mathfrak{L}_{jk}$. The formula (29) shows that the subspaces $\mathfrak{N}_{\alpha(j)}$ and $\mathfrak{N}_{\alpha(k)}$ are orthogonal with respect to the metric (17). Therefore the $\mathfrak{N}_{\alpha(k)}$ are invariant not only under the operators $L_i$, but also under the adjoint operators $L_i^*$.

Using the relation (30), we find that for $x, y \in \mathfrak{N}_{\alpha(k)}$

$$(L_i x, y) + (x, L_i y) = s(L_i(x \triangle y)) = \delta_{ik} s(x \triangle y) = \delta_{ik}(x, y).$$

This means that on $\mathfrak{N}_{\alpha(k)}$

$$L_i + L_i^* = \delta_{ik} E,$$

i.e., $L_i - \frac{1}{2}\delta_{ik}E$ is a skew-symmetric operator. Since the eigenvalues of $L_i$ are real, we finally obtain that

$$L_i = \frac{1}{2}\delta_{ik} E \quad \text{on } \mathfrak{N}_{\alpha(k)}. \tag{31}$$

Finally, it follows from the properties of a principal idempotent $u$ of $\mathfrak{L}_1$ and from the equation $v = e - u$, that

$$L_v = \frac{1}{2} E, \quad R_v = E \qquad \text{on } \mathfrak{N}, \tag{32}$$

$$L_v = R_v = 0 \quad \text{on } \mathfrak{L}_0. \tag{33}$$

We now put $e_m = v$, $\mathfrak{L}_{mm} = \{\lambda v\}$, $\mathfrak{L}_{km} = \mathfrak{N}_{\alpha(k)}$ $(k = 1, \cdots, m-1)$. It follows from (31), (32), (33) that the decomposition

$$\mathfrak{L} = \sum_{i \leq j} \mathfrak{L}_{ij} \quad (i, j = 1, \ldots, m)$$

is a normal decomposition of $\mathfrak{L}$.

From table (28) and by means of Lemma 1 we can easily deduce the following relations which hold for any normal decomposition of $\mathfrak{L}$:

$$\mathfrak{L}_{ij}\triangle\mathfrak{L}_{kl}=0, \quad \text{if} \quad j\neq k, l, \tag{34}$$

$$\mathfrak{L}_{ij}\triangle\mathfrak{L}_{jk}\subset\mathfrak{L}_{ik}, \tag{35}$$

$$\mathfrak{L}_{ij}\triangle\mathfrak{L}_{kj}\subset\mathfrak{L}_{ik} \quad \text{or} \quad \mathfrak{L}_{ki}. \tag{36}$$

It follows, in particular, that *the subspaces* $\mathfrak{L}_{ij}$ *are mutually orthogonal.*

When $a, b\in\mathfrak{L}_{ij}$ we have $a\Delta b=\lambda e_i$ and $(a, b)=s(a\Delta b)=\lambda s(e_i)$, whence $\lambda=(a, b)/s(e_i)$, so that

$$a\triangle b=\frac{(a, b)}{s(e_i)}e_i \qquad (a, b\in\mathfrak{L}_{ij}). \tag{37}$$

We observe that $s(e_i)=(e_i, e_i)>0$.

**Proposition 9.** *In the notation of Definition* 6, *every idempotent* $v$ *of* $\mathfrak{L}$ *can be expressed in the form*

$$v=e_{i_1}+\ldots+e_{i_k} \qquad (i_1<\ldots<i_k).$$

**Proof.** Let

$$v=\sum\lambda_i e_i+\sum_{i<j}a_{ij} \qquad (a_{ij}\in\mathfrak{L}_{ij}).$$

We assume that not all the $a_{ij}$ are zero. Let $i_0, j_0$ be such that $a_{i_0j_0}\neq 0$ but $a_{ij}=0$ for $j>j_0$ or $j=j_0$, $i>i_0$. Then the projection of the element $v\Delta v$ onto $\mathfrak{L}_{i_0j_0}$ is equal to

$$\sum\lambda_i e_i\triangle a_{i_0j_0}+a_{i_0j_0}\triangle\sum\lambda_i e_i=\left(\frac{\lambda_{i_0}+\lambda_{j_0}}{2}+\lambda_{j_0}\right)a_{i_0j_0}=\frac{\lambda_{i_0}+3\lambda_{j_0}}{2}a_{i_0j_0}.$$

Therefore, if $v$ is an idempotent, then

$$\frac{\lambda_{i_0}+3\lambda_{j_0}}{2}=1. \tag{38}$$

Further, the projection onto $\mathfrak{L}_{ii}$ gives

$$\lambda_i^2+\sum_{j>i}\frac{(a_{ij}, a_{ij})}{s(e_i)}=\lambda_i.$$

In particular, $\lambda_i\geq\lambda_i^2$, and so

$$0\leq\lambda_i\leq 1. \tag{39}$$

For $i=j_0$ we obtain $\lambda_{j_0}^2=\lambda_{j_0}$, i.e., $\lambda_{j_0}=0$ or 1. If we substitute these values of $\lambda_{j_0}$ into (38), we find that $\lambda_{i_0}=2$ or $-1$, which contradicts (39). Thus, $a_{ij}=0$ for all $i, j$. Clearly, in this case the coefficients $\lambda_i$ may be equal only to 0 or 1.

**Corollary.** *For every idempotent* $v \in \mathfrak{L}$ *we put*

$$\mathfrak{L}_v = \{x \in \mathfrak{L} : \ v \triangle x = x\}.$$

*The idempotents* $e_i$ *of the clan* $\mathfrak{L}$ *are the only idempotents for which*

$$\dim \mathfrak{L}_v = 1.$$

**Proof.** It is clear from (28) that if

$$v = e_{i_1} + \cdots + e_{i_k},$$

where $i_1 < \cdots < i_k$, then

$$\mathfrak{L}_v = \sum_{s \leq t} \mathfrak{L}_{i_s i_t}.$$

If $\dim \mathfrak{L}_v = 1$, then necessarily $k = 1$.

This Corollary gives an invariant characterization of the idempotents $e_i$, which will play a very important part in the proof of the uniqueness of the normal decomposition. But first of all we obtain from it the *invariance of the number m.*

Definition 7. The number $m$ in the normal decomposition of a clan $\mathfrak{L}$ with unit element (cf. Definition 6) is called the *rank of the clan* $\mathfrak{L}$ or the *rank of the convex homogeneous cone* $V(\mathfrak{L})$.

(The rank of a convex homogeneous cone could also have been defined as the dimension of its maximal connected commutative group of automorphisms, consisting of semisimple transformations with real eigenvalues.)

**Proposition 10.** *Let* $\mathfrak{L}$ *be a clan with a unit element and let*

$$\mathfrak{L} = \sum_{1 \leq i \leq j \leq m} \mathfrak{L}_{ij}^{(p)} \qquad (p = 1, 2)$$

*be two normal decompositions. Then there exists a permutation* $i \to \tilde{i}$ *of the set of indices* $1, \cdots, m$ *such that*

1) *if* $i \leq j$, $\tilde{i} \leq \tilde{j}$, *then* $\mathfrak{L}^{(2)}_{ij} = \mathfrak{L}^{(1)}_{\tilde{i}\tilde{j}}$;

2) *if* $i < j$, *but* $\tilde{i} > \tilde{j}$, *then* $\mathfrak{L}^{(2)}_{ij} = \mathfrak{L}^{(1)}_{\tilde{j}\tilde{i}} = 0$.

**Proof.** It follows from the Corollary to Proposition 9 that there exists a permutation $i \to \tilde{i}$, such that $e^{(2)}_i = e^{(1)}_{\tilde{i}}$. Clearly, $\mathfrak{L}^{(2)}_{ii} = \mathfrak{L}^{(1)}_{\tilde{i}\tilde{i}}$. Further, for $i < j$

$$\mathfrak{L}^{(2)}_{ij} = \left\{x \in \mathfrak{L} : e^{(2)}_i \triangle x = e^{(2)}_j \triangle x = \frac{1}{2} x\right\} = \left\{x \in \mathfrak{L} : e^{(1)}_{\tilde{i}} \triangle x = e^{(1)}_{\tilde{j}} \triangle x = \frac{1}{2} x\right\},$$

which implies that $\mathfrak{L}^{(2)}_{ij} = \mathfrak{L}^{(1)}_{\tilde{i}\tilde{j}}$ if $\tilde{i} < \tilde{j}$ and $\mathfrak{L}^{(2)}_{ij} = \mathfrak{L}^{(1)}_{\tilde{j}\tilde{i}}$ if $\tilde{i} > \tilde{j}$. In the latter case we have

$$\mathfrak{L}^{(2)}_{ij} \triangle \mathfrak{L}^{(2)}_{ij} = \mathfrak{L}^{(1)}_{\tilde{j}\tilde{i}} \triangle \mathfrak{L}^{(1)}_{\tilde{j}\tilde{i}} \subset \mathfrak{L}^{(1)}_{\tilde{j}\tilde{j}} = \mathfrak{L}^{(2)}_{jj};$$

but, on the other hand,

$$\mathfrak{L}^{(2)}_{ij} \triangle \mathfrak{L}^{(2)}_{ij} \subset \mathfrak{L}^{(2)}_{ii}.$$

Therefore, $\mathfrak{L}_{ij}^{(2)} \Delta \mathfrak{L}_{ij}^{(2)} = 0$ and $\mathfrak{L}_{ij}^{(2)} = 0$ (cf. formula (37)).

We now establish some properties of clans which will be important in Chapter III. We use the notation of Definition 6 and we take $a_{ij}, b_{ij}, \cdots$ to be arbitrary elements of $\mathfrak{L}_{ij}$.

Since $\mathfrak{L}$ is left-symmetric, it follows that if $i < j < k$,

$$(a_{ij},\ b_{jk} \triangle c_{ik}) = s\,(L_{a_{ij}} L_{b_{jk}} c_{ik}) = s\,((L_{b_{jk}} L_{a_{ij}} + L_{[a_{ij} \triangle b_{jk}]})\, c_{ik}) = s\,(L_{a_{ij} \triangle b_{jk}} c_{ik}),$$

i.e.,

$$(a_{ij},\ b_{jk} \triangle c_{ik}) = (a_{ij} \triangle b_{jk},\ c_{ik}). \tag{40}$$

In the same way we derive the formula

$$(a_{ij},\ c_{ik} \triangle b_{jk}) = (a_{ij} \triangle b_{jk},\ c_{ik}). \tag{41}$$

The formulae (40) and (41) show that the products of the form $b_{jk} \Delta c_{ik}$ and $c_{ik} \Delta b_{jk}$ are uniquely determined once we are given the products of the form $a_{ij} \Delta b_{jk}$.

Further

$$\begin{aligned}(a_{ij} \triangle b_{jk}) \triangle (a_{ij} \triangle b_{jk}) &= [(a_{ij} \triangle b_{jk}) \triangle a_{ij}] \triangle b_{jk} \\ &+ a_{ij} \triangle ((a_{ij} \triangle b_{jk}) \triangle b_{jk}) = a_{ij} \triangle ([a_{ij} \triangle b_{jk}] \triangle b_{jk}) \\ &= a_{ij} \triangle (a_{ij} \triangle (b_{jk} \triangle b_{jk})) - a_{ij} \triangle (b_{jk} \triangle (a_{ij} \triangle b_{jk})) \\ &= \frac{(b_{jk},\ b_{jk})}{s\,(e_j)}\, a_{ij} \triangle (a_{ij} \triangle e_j) - a_{ij} \triangle (b_{jk} \triangle (a_{ij} \triangle b_{jk})) \\ &= \frac{(a_{ij},\ a_{ij})\,(b_{jk},\ b_{jk})}{s\,(e_i)\, s\,(e_j)}\, e_i - a_{ij} \triangle (b_{jk} \triangle (a_{ij} \triangle b_{jk})),\end{aligned}$$

whence, by (40),

$$\begin{aligned}(a_{ij} \triangle b_{jk},\ a_{ij} \triangle b_{jk}) &= \frac{(a_{ij},\ a_{ij})\,(b_{jk},\ b_{jk})}{s\,(e_j)} - (a_{ij},\ b_{jk} \triangle (a_{ij} \triangle b_{jk})) \\ &= \frac{(a_{ij},\ a_{ij})\,(b_{jk},\ b_{jk})}{s\,(e_j)} - (a_{ij} \triangle b_{jk},\ a_{ij} \triangle b_{ik}),\end{aligned}$$

so that, if $i < j < k$,

$$(a_{ij} \triangle b_{jk},\ a_{ij} \triangle b_{jk}) = \frac{1}{2s\,(e_j)}\,(a_{ij},\ a_{ij})\,(b_{jk},\ b_{jk}). \tag{42}$$

The next formulae are the identity (5) written out in two special cases when, by (34), it takes a particularly simple form:

$$[a_{ij} \triangle b_{jk} \triangle c_{kl}] = 0 \qquad (j < k), \tag{43}$$

$$[a_{ij} \triangle b_{jl} \triangle c_{kl}] = 0 \qquad (j \neq k,\ l). \tag{44}$$

# CHAPTER III

# MATRIX CALCULUS

## §1. $T$-algebras

In this section we shall construct a formalism which will be convenient for describing and studying convex homogeneous cones.

Definition 1. A *matrix algebra of rank* $m$ is an algebra $\mathfrak{A}$ bigraded by subspaces $\mathfrak{A}_{ij}$ $(i, j = 1, \cdots, m)$ such that

$$\mathfrak{A}_{ij}\mathfrak{A}_{jk} \subset \mathfrak{A}_{ik}$$

and, for $j \neq 1$,

$$\mathfrak{A}_{ij}\mathfrak{A}_{lk} = 0.$$

It is convenient to represent the element $a$ of $\mathfrak{A}$ by the matrix $(a_{ij})$, where the $a_{ij}$ are the projections of $a$ onto the $\mathfrak{A}_{ij}$. Here the matrix multiplication is carried out according to the usual rules.

Definition 2. An *involution of a matrix algebra* $\mathfrak{A}$ is a linear mapping $^*$ of $\mathfrak{A}$ onto itself that satisfies the following conditions:

1) $a^{**} = a$;
2) $(ab)^* = b^*a^*$;
3) $\mathfrak{A}_{ij}^* \subset \mathfrak{A}_{ji}$.

In matrix notation an involution is "transposition and conjugation":

$$(a^*)_{ij} = a_{ji}^*.$$

Let $\mathfrak{A}$ be a matrix algebra. We put

$$\mathfrak{T} = \sum_{i \leq j} \mathfrak{A}_{ij}. \tag{1}$$

The subspace $\mathfrak{T}$ is a subalgebra of $\mathfrak{A}$; its elements are written in the form of upper triangular matrices. If we are given an involution in the algebra $\mathfrak{A}$, then we can define the subspace

$$\mathfrak{X} = \{x \in \mathfrak{A}\colon\ x^* = x\} \tag{2}$$

of "Hermitian matrices" and the subspace

$$\mathfrak{K} = \{k \in \mathfrak{A}\colon\ k^* = -k\} \tag{3}$$

of "skew-Hermitian matrices." Clearly $\mathfrak{A} = \mathfrak{X} \dotplus \mathfrak{K}$.

We shall also use the following notation:

$$[ab] = ab - ba, \tag{4}$$

$$[abc] = a(bc) - (ab)c, \tag{5}$$

$$n_{ij} = \dim \mathfrak{A}_{ij}. \tag{6}$$

In a matrix algebra with involution

$$n_{ij} = n_{ji}. \tag{7}$$

In any matrix algebra $\mathfrak{A}$ the subspaces $\mathfrak{A}_{ii}$ are subalgebras. It may happen that they are all isomorphic to the algebra $R$ of real numbers. The unique isomorphism of $\mathfrak{A}_{ii}$ onto $R$ will then be denoted by $\rho$ and the unit element of $\mathfrak{A}_{ii}$ by $e_i$. Further, the "trace" of the matrix $a$ is defined as follows:

$$\operatorname{Sp} a = \sum n_i \varrho(a_{ii}). \tag{8}$$

where

$$n_i = 1 + \frac{1}{2} \sum_{s \neq i} n_{is}.\text{[1]} \tag{9}$$

**Definition 3.** A matrix algebra $\mathfrak{A}$ with an involution * is called a *T-algebra* if the following conditions are satisfied (cf. the notation adopted above):

(I) all the subalgebras $\mathfrak{A}_{ii}$ are isomorphic to the algebra $R$ of real numbers;

(II) for every $a_{ij} \in \mathfrak{A}_{ij}$

$$e_i a_{ij} = a_{ij} e_j = a_{ij};$$

(III) $\operatorname{Sp}[ab] = 0$;

(IV) $\operatorname{Sp}[abc] = 0$;

(V) $\operatorname{Sp} aa^* > 0$, if $a \neq 0$;

(VI) for any $t, u, w \in \mathfrak{T}$

$$[tuw] = 0;$$

(VII) for any $t, u \in \mathfrak{T}$

$$[tuu^*] = 0.$$

By applying the involution * to the identity (VI) we obtain the equivalent condition

(VI′) for any $t, u, w \in \mathfrak{T}$

$$[t^* u^* w^*] = 0.$$

Further, the condition (VII) can be written in the following polarized form:

(VII′) for any $t, u, w \in \mathfrak{T}$

$$[tuw^*] + [twu^*] = 0,$$

whence, if we apply the involution, we obtain the condition

(VII″) for any $t, u, w \in \mathfrak{T}$

$$[tu^* w^*] + [ut^* w^*] = 0.$$

---

[1] This definition of the numbers $n_i$ is essential only for the assertion of Lemma 2, used in §§4, 5. Everywhere else the $n_i$ may be taken to be fixed positive numbers.

Finally, we note that in a $T$-algebra

$$\operatorname{Sp} a^* = \operatorname{Sp} a, \tag{10}$$

because for fixed $i$ the involution * induces an isomorphic and, therefore, identical mapping of $\mathfrak{A}_{ii} \simeq R$ onto itself.

Let $\mathfrak{A}$ be a $T$-algebra of rank $m$. We shall refer to the permutation $i \to \tilde{i}$ of the set $\{1, \cdots, m\}$ as *admissible* if from $1 < j$, $\tilde{i} > \tilde{j}$ it follows that $n_{ij} = 0$.

Definition 4. An *inessential change in the grading* of a $T$-algebra $\mathfrak{A}$ is a change in its grading consisting in the replacement of each subspace $\mathfrak{A}_{ij}$ by $\mathfrak{A}_{\tilde{i}\tilde{j}}$, where $i \to \tilde{i}$ is an admissible permutation.

The point of the conditions leading to an inessential change in the grading is that the subspace $\mathfrak{T}$ of triangular matrices remains unaltered.

Definition 5. A mapping of a $T$-algebra $\mathfrak{A}$ onto a $T$-algebra $\tilde{\mathfrak{A}}$ is said to be *isomorphic* if, after an inessential change in the grading of $\mathfrak{A}$ (or, equally, of $\tilde{\mathfrak{A}}$) it becomes an isomorphism of the bigraded algebras with involution. Two $T$-algebras are said to be *isomorphic* if there exists an isomorphic mapping of one of them onto the other.

Let $\mathfrak{A}$ be a $T$-algebra of rank $m$. For each matrix $a = (a_{ij}) \in \mathfrak{A}$ we put

$$\hat{a} = \frac{1}{2} \sum a_{ii} + \sum_{i<j} a_{ij}, \tag{11}$$

$$\check{a} = \frac{1}{2} \sum a_{ii} + \sum_{i>j} a_{ij}. \tag{12}$$

Clearly, $\hat{a}$ is an upper triangular matrix, $\check{a}$ is a lower triangular matrix and

$$a = \hat{a} + \check{a}. \tag{13}$$

It is also clear that

$$\widehat{a^*} = (\check{a})^*, \quad \check{(a^*)} = (\hat{a})^*. \tag{14}$$

In particular, for a Hermitian matrix $x$,

$$\check{x} = (\hat{x})^*. \tag{15}$$

We define a bilinear operation $\Delta$ in the space $\mathfrak{A}$ by the formula

$$a \triangle b = \hat{a}b + b\check{a}. \tag{16}$$

Lemma 1. *For any* $a, b \in \mathfrak{A}$

$$\widehat{[a \triangle b]} = [\hat{a}\, \hat{b}], \quad \underbrace{[a \triangle b]}_{\vee} = -[\check{a}\, \check{b}].$$

**Proof.** In view of (13)

$$[a \triangle b] = \hat{a}b + b\check{a} - \hat{b}a - a\check{b}$$
$$= \hat{a}\hat{b} + \hat{a}\check{b} + \hat{b}\check{a} + \check{b}\check{a} - \hat{b}\hat{a} - \hat{b}\check{a} - \hat{a}\check{b} - \check{a}\check{b} = [\hat{a}\hat{b}] - [\check{a}\check{b}].$$

It is easy to see that $[\hat{a}\hat{b}]$ and $[\check{a}\check{b}]$ are upper and lower triangular matrices, respectively, with zeros on the diagonals. This implies the result.

The space $\mathfrak{X}$ of Hermitian matrices of $\mathfrak{A}$ is closed under the operation $\Lambda$. In fact, for any Hermitian matrices $x, y$

$$(x \triangle y)^* = (\hat{x}y + y\check{x})^* = y\check{x} + \hat{x}y = x \triangle y.$$

**Lemma 2.** *For any* $x \in \mathfrak{X}$ *the operator*

$$L_x\colon\ y \to x \triangle y \quad (y \in \mathfrak{X})$$

*in* $\mathfrak{X}$ *has only real eigenvalues. Its trace is equal to* Sp $x$.

**Proof.** For $i \leq j$ we put

$$\mathfrak{X}_{ij} = \mathfrak{X} \cap (\mathfrak{A}_{ij} + \mathfrak{A}_{ji}). \tag{17}$$

The space $\mathfrak{X}$ splits into the direct sum of the subspaces $\mathfrak{X}_{ij}$ $(i, j = 1, \cdots, m)$, where

$$\dim \mathfrak{X}_{ij} = \begin{cases} n_{ij} & \text{if } \ i < j, \\ 1 & \text{if } \ i \leqslant j. \end{cases}$$

Further, let

$$\mathfrak{X}^{ij} = \sum_{k<i} \mathfrak{X}_{kl} + \sum_{l \leqslant j} \mathfrak{X}_{il} \quad (i \leqslant j). \tag{18}$$

If $i < j$, then $\mathfrak{X}_{ij}$ is naturally isomorphic to the factor space $\mathfrak{X}^{ij} / \mathfrak{X}^{i,j-1}$ and $\mathfrak{X}_{ii}$ to the factor space $\mathfrak{X}^{ii} / \mathfrak{X}^{i-1,m}$ (if we take $\mathfrak{X}^{0m} = 0$). If $y \in \mathfrak{X}_{ij}$, then

$$L_x y = \hat{x}y + y\check{x} = \frac{1}{2}(\varrho(x_{ii}) + \varrho(x_{jj}))(y_{ij} + y_{ji})$$
$$+ \sum_{s<i} x_{si}y_{ij} + \sum_{s<j} y_{ij}x_{js} + \sum_{s<j} x_{sj}y_{ji} + \sum_{s<i} y_{ji}x_{is}.$$

Here it is clear that

$$L_x y \equiv \frac{1}{2}(\varrho(x_{ii}) + \varrho(x_{jj}))\, y \begin{cases} (\operatorname{mod} \mathfrak{X}^{i,j-1}) & \text{if} \quad i < j, \\ (\operatorname{mod} \mathfrak{X}^{i-1,m}) & \text{if} \quad i = j. \end{cases}$$

Therefore, the eigenvalues of $L_x$ are of the form $\frac{1}{2}(\rho(x_{ii}) + \rho(x_{jj}))$. Their sum, with the corresponding multiplicities, is equal to

$$\sum \varrho(x_{ii}) + \frac{1}{2}\sum_{i<j} n_{ij}(\varrho(x_{ii}) + \varrho(x_{jj})) = \operatorname{Sp} x.$$

This completes the proof of the lemma.

## §2. The convex cone connected with a $T$-algebra

The axiom (VI) in Definition 3 shows that the triangular matrices of $\mathfrak{A}$ form an associative subalgebra of it. If the triangular matrix $t$ is such that $t_{ii} \neq 0$ for all $i$, then it is not a divisor of zero in this subalgebra. Therefore the set

$$\mathcal{T}(\mathfrak{A}) = \{t \in \mathfrak{T}: t_{ii} > 0 \ (i = 1, \ldots, m)\} \tag{19}$$

is open in $\mathfrak{T}$ and is a connected Lie group. Its Lie algebra $T(\mathfrak{A})$ can be identified with $\mathfrak{T}$ when the commutation operation is defined by the formula

$$[t, u] = [tu]. \tag{20}$$

We consider the mapping

$$F: t \longrightarrow tt^* \in \mathfrak{X} \quad (t \in \mathfrak{T}). \tag{21}$$

Let $e$ denote the unit matrix. The principal linear part of $F$ at $e$ is of the form

$$dF: t \longrightarrow t + t^*$$

and is an isomorphism of the linear space $\mathfrak{T}$ onto $\mathfrak{X}$. Therefore, the image of $\mathcal{T}(\mathfrak{A})$ under $F$ contains a neighborhood of the matrix $F(e) = e$ in $\mathfrak{X}$.

We shall show in §3 that every Hermitian matrix can be expressed in the form $tt^*$ in precisely one way, where $t \in \mathcal{T}(\mathfrak{A})$. In other words, the set

$$V(\mathfrak{A}) = \{tt^* : t \in \mathcal{T}(\mathfrak{A})\} \tag{22}$$

is a one-to-one image of $\mathcal{T}(\mathfrak{A})$ under $F$. The transformations

$$\pi(w) : uu^* \longrightarrow (wu)(u^*w^*) \qquad (u, w \in \mathcal{T}(\mathfrak{A})) \tag{23}$$

of $V(\mathfrak{A})$ correspond to the left translations of $\mathcal{T}(\mathfrak{A})$. Differentiating with respect to $w$ and putting $w = e$, $dw = t \in \mathfrak{T}$, we find the corresponding infinitesimal transformation:

$$d\pi(t) : uu^* \longrightarrow (tu)u^* + u(u^*t^*).$$

The axiom (VII) for a $T$-algebra shows that the transformation $d\pi(t)$ is the restriction to $V(\mathfrak{A})$ of the infinitesimal linear transformation

$$D_t : y \longrightarrow ty + yt^* \quad (y \in \mathfrak{X}) \tag{24}$$

of $\mathfrak{X}$. Therefore the transformations $\pi(w)$, $w \in \mathcal{T}(\mathfrak{A})$, are also the restrictions to $V(\mathfrak{A})$ of certain linear transformations of $\mathfrak{X}$. These transformations clearly act transitively on $V(\mathfrak{A})$.

If $x = t + t^*$, $t \in \mathfrak{T}$, then

$$\hat{x} = t, \quad \check{x} = t^*$$

and the transformation $D_t$, defined by (24), coincides with

$$L_x : y \longrightarrow x \Delta y.$$

Therefore, the transformations $L_x$, $x \in \mathfrak{X}$, form a Lie algebra. Suppose that $x, y \in \mathfrak{X}$ and that

$$[L_x, L_y] = L_z \quad (z \in \mathfrak{X}).$$

Applying the last equation to the matrix $e$ we find that

$$z = L_z e = [L_x, L_y]\, e = [x \triangle y].$$

Thus,

$$[L_x, L_y] = L_{[x \triangle y]},$$

i.e., the operation $\Delta$ in $\mathfrak{X}$ satisfies the conditions for left symmetry (cf. §1 of Chapter II).

By $\mathfrak{L}(\mathfrak{A})$ we denote the algebra defined on $\mathfrak{X}$ by the operation $\Delta$, and we shall show that $\mathfrak{L}(\mathfrak{A})$ is a clan with a unit element.

We have already proved that $\mathfrak{L}(\mathfrak{A})$ is left-symmetric. We have also proved (Lemma 2) that the operators of left multiplication in $\mathfrak{L}(\mathfrak{A})$ have only real eigenvalues. The unit matrix is the unit element in $\mathfrak{L}(\mathfrak{A})$. Thus, it remains to show that the left-symmetric algebra $\mathfrak{L}(\mathfrak{A})$ is compact. For this purpose we put

$$s(x) = \operatorname{Sp} x \quad (x \in \mathfrak{X}).$$

By Lemma 1, for any $x, y \in \mathfrak{X}$,

$$s([x \triangle y]) = s([\hat{x}\, \hat{y}]) - s([\check{x}\, \check{y}]) = 0.$$

Further, by axioms (III) and (V) for a $T$-algebra

$$s(x \triangle x) = s(\hat{x}x + x\check{x}) = s(\hat{x}x + \check{x}x) = s(x^2) > 0,$$

if $x \neq 0$.

Clearly

$$V(\mathfrak{A}) = V(\mathfrak{L}(\mathfrak{A}))$$

(cf. §2 of Chapter II). Therefore $V(\mathfrak{A})$ is a convex homogeneous cone.

An isomorphic mapping of $T$-algebras preserves the property of matrices being triangular and therefore commutes with the operation $\hat{}$. Therefore isomorphic $T$-algebras correspond to isomorphic left-symmetric algebras and hence to isomorphic convex cones.

**Proposition 1.** *For every $T$-algebra $\mathfrak{A}$ the set $V(\mathfrak{A})$ (cf. 22)) is a convex homogeneous cone in which, by* (23), *the group $\mathfrak{T}(\mathfrak{A})$ acts linearly and transitively* (*cf.* (19)). *Isomorphic $T$-algebras correspond to isomorphic convex cones.*

## §3. The equation of the cone $V(\mathfrak{A})$

In the $T$-algebra $\mathfrak{A}$ of rank $m$ we consider the subspaces

$$\mathfrak{A}^{(k)} = \sum_{i,j=1}^{k} \mathfrak{A}_{ij} \qquad (k = 1, \ldots, m).$$

With every Hermitian matrix $x \in \mathfrak{A}$ we associate a sequence of matrices $x^{(k)} \in \mathfrak{A}^{(k)}$ $(k = 1, \cdots, m)$ as follows:

$$x^{(m)} = x,$$

$$x^{(k-1)} = \sum_{i,j=1}^{k-1} (\varrho(x_{kk}^{(k)})\, x_{ij}^{(k)} - x_{ik}^{(k)} x_{kj}^{(k)}).$$

We can say that the matrix $x^{(k-1)}$ is formed from the second-order "minors" of $x^{(k)}$. We put

$$p_k(x) = \varrho(x_{kk}^{(k)}) \qquad (k = 1, \ldots, m). \tag{25}$$

It is easy to see that $p_k(x)$ is a homogeneous polynomial of degree $2^{m-k}$ in the coordinates of the vector $x \in \mathfrak{X}$.

Lemma 3. *If $x = tt^*$, where $t \in \mathfrak{T}$, then*

$$x_{ij}^{(k)} = \Big(\prod_{s>k} p_s(x)\Big) \sum_{l=1}^{k} t_{il} t_{jl}^*.$$

Proof. We prove the lemma by induction, passing step-by-step to smaller values of $k$. We assume that for some $k$ the assertion of the lemma has been proved. Then, for any $i, j \le k-1$,

$$\begin{aligned} x_{ij}^{(k-1)} &= \varrho(x_{kk}^{(k)})\, x_{ij}^{(k)} - x_{ik}^{(k)} x_{kj}^{(k)} \\ &= \Big(\prod_{s\geqslant k} p_s(x)\Big) \sum_{l=1}^{k} t_{il} t_{jl}^* - \Big(\prod_{s>k} p_s(x)\Big)^2 (\varrho(t_{kk}))^2\, t_{ik} t_{jk}^* \\ &= \Big(\prod_{s\geqslant k} p_s(x)\Big) \Big(\sum_{l=1}^{k} t_{il} t_{jl}^* - t_{ik} t_{jk}^*\Big) = \Big(\prod_{s\geqslant k} p_s(x)\Big) \sum_{l=1}^{k-1} t_{il} t_{jl}^*, \end{aligned}$$

since

$$\Big(\prod_{s>k} p_s(x)\Big) (\varrho(t_{kk}))^2 = x_{kk}^{(k)} = p_k(x).$$

Proposition 2. *Let $\mathfrak{A}$ be a $T$-algebra of rank $m$, and $\mathfrak{X}$ the space of Hermitian matrices of $\mathfrak{A}$. The cone $V(\mathfrak{A})$ is distinguished in $\mathfrak{X}$ by the inequalities*

$$p_k(x) > 0 \quad (k = 1, \cdots m). \tag{26}$$

*A Hermitian matrix $x \in V(\mathfrak{A})$ can be written in exactly one way in the form $tt^*$, where $t \in \mathfrak{T}(\mathfrak{A})$.*

**Proof.** Let $x \in V(\mathfrak{A})$. Then $x = tt^*$, where $t \in \mathfrak{T}(\mathfrak{A})$. It follows from Lemma 3 that

$$p_k(x) = \varrho(x_{kk}^{(k)}) = \Big(\prod_{s>k} p_s(x)\Big)(\varrho(t_{kk}))^2$$

and so

$$\frac{p_k(x)}{\prod\limits_{s>k} p_s(x)} > 0.$$

Hence we easily deduce by induction on $k$ (from larger values to smaller ones) that

$$p_k(x) > 0 \qquad (k = 1, \ldots, m).$$

Further

$$t_{kk} = \varrho(t_{kk})\, e_k = \sqrt{\frac{p_k(x)}{\prod\limits_{s>k} p_s(x)}}\, e_k, \tag{27}$$

and, by Lemma 3, for any $i \leq k$

$$x_{ik}^{(k)} = \Big(\prod_{s>k} p_s(x)\Big)\varrho(t_{kk})\, t_{ik},$$

whence

$$t_{ik} = \frac{x_{ik}^{(k)}}{\sqrt{\prod\limits_{s\geqslant k} p_s(x)}} \qquad (i \leqslant k). \tag{28}$$

This shows that $t$ is uniquely determined by $x$.

Conversely, if a Hermitian matrix $x$ satisfies the inequalities (26), then we can construct the triangular matrix $t \in \mathfrak{T}(\mathfrak{A})$, defining its elements by (28). We consider the Hermitian matrix $\widetilde{x} = tt^*$. Arguing as in the first part of the proof, we find that

$$t_{ik} = \frac{\widetilde{x}_{ik}^{(k)}}{\sqrt{\prod\limits_{s\geqslant k} p_s(\widetilde{x})}} \qquad (i \leqslant k).$$

Therefore

$$\frac{\widetilde{x}_{ik}^{(k)}}{\sqrt{\prod\limits_{s\geqslant k} p_s(\widetilde{x})}} = \frac{x_{ik}^{(k)}}{\sqrt{\prod\limits_{s\geqslant k} p_s(x)}} \qquad (i \leqslant k). \tag{29}$$

In particular, if $i = k$, we find that

$$\frac{p_k(\widetilde{x})}{\prod\limits_{s>k} p_s(\widetilde{x})} = \frac{p_k(x)}{\prod\limits_{s>k} p_s(x)},$$

which implies that for any $k$

$$p_k(\widetilde{x}) = p_k(x).$$

It then follows from (29) that, for $i \leq j$,

$$\widetilde{x}_{ij}^{(j)} = x_{ij}^{(j)}. \tag{30}$$

Suppose that $i \leq j$ and that for some $k > j$

$$\widetilde{x}_{ij}^{(k-1)} = x_{ij}^{(k-1)}$$

Then

$$\widetilde{x}_{ij}^{(k)} = \frac{\widetilde{x}_{ij}^{(k-1)} + \widetilde{x}_{ik}^{(k)} (\widetilde{x}_{jk}^{(k)})^*}{p_k(\widetilde{x})} = \frac{x_{ij}^{(k-1)} + x_{ik}^{(k)} (x_{jk}^{(k)})^*}{p_k(x)} = x_{ij}^{(k)}.$$

This argument allows us to raise the upper index in (30) one step at a time till we reach $m$ and obtain the equation

$$\widetilde{x}_{ij} = x_{ij}.$$

Thus, $\widetilde{x} = x$ and $x \in V(\mathfrak{A})$.

Proposition 2 (more precisely, its first part) may be called Sylvester's criterion for the cone $V(\mathfrak{A})$.

## §4. The calculation of the invariant measure

We consider the subgroup

$$\mathcal{T}_0 = \{w \in \mathcal{T}(\mathfrak{A}) : \prod (\varrho(w_{ii}))^{n_i} = 1\}$$

of $\mathfrak{T}(\mathfrak{A})$. Its Lie algebra $T_0$ is formed by the triangular matrices $t$ for which

$$\sum n_i \varrho(t_{ii}) = \operatorname{Sp} t = 0.$$

By Lemma 2, for every matrix $t \in T_0$

$$\operatorname{Sp} D_t = \operatorname{Sp} L_{t+t^*} = \operatorname{Sp}(t + t^*) = 0$$

(cf. (24)). Therefore, for every matrix $w \in \mathfrak{T}_0$

$$\det \pi(w) = 1$$

(cf. (23)).

For every $\lambda > 0$

$$\pi(\lambda e)\, x = \lambda^2 x \qquad (x \in \mathfrak{X}),$$

so that

$$\det \pi(\lambda e) = \lambda^{2n},$$

where

$$n = \dim \mathfrak{X} = \sum n_i. \tag{31}$$

Every matrix $t \in \mathfrak{T}(\mathfrak{A})$ can be expressed in the form $t = \lambda w$, where $\lambda > 0$, $w \in \mathfrak{T}_0$. The number $\lambda$ is determined from the condition

We have

$$\prod (\varrho (t_{ii}))^{n_i} = \prod (\lambda \varrho (w_{ii}))^{n_i} = \lambda^n.$$

$$\det \pi (\lambda w) = \det \pi (\lambda e) \det \pi (w) = \lambda^{2n} = \prod (\varrho (t_{ii}))^{2n_i}.$$

Thus,

$$\det \pi (t) = \prod (\varrho (t_{ii}))^{2n_i}. \tag{32}$$

Now we have everything ready to calculate the density $\phi(x)$ of the invariant measure in the cone $V(\mathfrak{A})$. If we take

$$\phi(e) = 1,$$

then, for every $t \in \mathfrak{T}(\mathfrak{A})$,

$$\varphi (tt^*) = \varphi (\pi (t) e) = (\det \pi (t))^{-1}.$$

Let $x = tt^* \in V(\mathfrak{A})$. By formula (27)

$$\varrho (t_{ii}) = \sqrt{\frac{p_i (x)}{\prod\limits_{s>i} p_s (x)}}.$$

Therefore

$$\det \pi (t) = \prod_i \left( \frac{p_i (x)}{\prod\limits_{s>i} p_s (x)} \right)^{n_i} = \prod (p_i (x))^{n_i - n_{i-1} - \ldots - n_1}$$

and

$$\varphi (x) = \prod (p_i (x))^{n_1 + \ldots + n_{i-1} - n_i}. \tag{33}$$

Since the numbers $2n_i$ are integers, $\phi^2$ is a rational function; in some cases $\phi$ itself is rational.

## §5. The calculation of the algebra of connectedness

In §3 of Chapter I we defined a Riemannian metric on a convex cone. In §1 of Chapter II we calculated the first few terms in the Taylor series expansion of $\ln \phi$ in the neighborhood of $x_0$ that was chosen for the construction of the clan; in particular, we found the values of the metric tensor $g$ and of the object of the linear connection $\Gamma$ at this point. We shall now apply these results to the convex cone $V(\mathfrak{A})$ corresponding to the $T$-algebra $\mathfrak{A}$. In this case the role of $x_0$ will be taken by the unit matrix $e$.

For brevity we put

$$(g(e))(x, y) = (x, y) \qquad (x, y \in \mathfrak{X}).$$

The formula (10) of Chapter II yields

$$(x, y) = s(x \Delta y),$$

where, by virtue of Lemma 2,

$$s(z) = \operatorname{Sp} L_z = \operatorname{Sp} z \quad (z \in \mathfrak{X}).$$

Since

$$\operatorname{Sp}(x \triangle y) = \operatorname{Sp}(\hat{x}y + y\check{x}) = \operatorname{Sp}(\hat{x}y + \check{x}y) = \operatorname{Sp} xy,$$

it follows that

$$(x, y) = \operatorname{Sp} xy. \tag{34}$$

The algebra of connectedness of the cone $V(\mathfrak{A})$ at $e$ is defined by specifying the multiplication operation $\square$ in $\mathfrak{X}$:

$$(x \square y)^i = -\Gamma^i_{jk}(e)\, x^j y^k$$

(formula (5) of Chapter I). By formula (4) of Chapter I

$$\Gamma^i_{jk}(e) = \frac{1}{2} g^{il}(e)\, \partial_{jkl} \ln \varphi(e).$$

We put

$$Q(x, y, z) = (\partial_{jkl} \ln \varphi(e))\, x^j y^k z^l \qquad (x, y, z \in \mathfrak{X}).$$

The operation $\square$ can then be determined from the condition

$$(x \square y, z) = -\frac{1}{2} Q(x, y, z). \tag{35}$$

In fact,

$$(x \square y, z) = -g_{il}(e)\, \Gamma^i_{jk}(e)\, x^j y^k z^l = -\frac{1}{2} (\partial_{jkl} \ln \varphi(e))\, x^j y^k z^l.$$

The cubic form $d^3 \ln \phi(e)$ corresponds to the symmetric trilinear form $Q$. By formula (11) of Chapter II

$$Q(x, x, x) = (d^3 \ln \varphi(e))(x) = -2 \operatorname{Sp}(x \triangle (x \triangle x)). \tag{36}$$

Theorem 3. *Let $\mathfrak{A}$ be a $T$-algebra and $V(\mathfrak{A})$ the corresponding convex cone in the space $X$ of Hermitian matrices. The structure of the algebra of connectedness of the cone $V(\mathfrak{A})$ at the point $e \in V(\mathfrak{A})$ is given in $\mathfrak{X}$ by the formula*

$$x \square y = \frac{1}{2}(xy + yx).$$

Proof. By virtue of (35) it suffices to show that for any $x, y, z \in \mathfrak{X}$

$$(xy + yx, z) = -Q(x, y, z).$$

This relation will be proved if we show that the trilinear form

$$R(x, y, z) = (xy + yx, z)$$

is symmetric and that

$$R(x, x, x) = 2 \operatorname{Sp}(x \Delta (x \Delta x)) \tag{37}$$

(cf. (36)).

Clearly, the form $R$ is symmetric in the first two arguments. Further, by virtue of (34) and axioms (III) and (IV) for $T$-algebras

$$R(x, z, y) = (xz + zx, y) = \mathrm{Sp}((xz)y + (zx)y)$$
$$= \mathrm{Sp}((yx)z + (xy)z) = (xy + yx, z) = R(x, y, z).$$

This implies that $R$ does not change under any permutation of the arguments.

We now prove the relation (37). Since $\mathrm{Sp}(x \Delta y) = \mathrm{Sp}\, xy$ for any $x, y \in \mathfrak{X}$, it follows that

$$\mathrm{Sp}(x \triangle (x \triangle x)) = \mathrm{Sp}\, x(\hat{x}x + \check{x}x)$$
$$= \mathrm{Sp}\, x^2(\hat{x} + \check{x}) = \mathrm{Sp}(x^2)x = (x^2, x) = \frac{1}{2} R(x, x, x).$$

## §6. The adjoint cone

Let $\mathfrak{A}$ be a $T$-algebra of rank $m$. We consider the matrix algebra with involution $\mathfrak{A}'$ which differs from $\mathfrak{A}$ only in its grading, and we put

$$\mathfrak{A}'_{ij} = \mathfrak{A}_{m+1-i,\ m+1-j} \qquad (i, j = 1, \ldots, m).$$

If $\mathfrak{T}'$ is the subspace of upper triangular matrices of $\mathfrak{A}'$, then

$$\mathfrak{T}' = \mathfrak{T}^*. \tag{38}$$

Therefore the axioms (VI$'$) and (VII$''$) for $\mathfrak{A}$ correspond to the axioms (VI) and (VII$'$) for $\mathfrak{A}'$. It is easy to see that the trace of a matrix does not depend on whether it is considered as an element of $\mathfrak{A}$ or of $\mathfrak{A}'$. All this implies that $\mathfrak{A}'$ is also a $T$-algebra.

The cone $V(\mathfrak{A}')$ is the orbit in $\mathfrak{X}' = \mathfrak{X}$ of the linear group $\mathfrak{I}(\mathfrak{A}')$ generated by the infinitesimal transformations

$$D'_t:\ x \to t^*x + xt,\ t \in \mathfrak{T}$$

(cf. (24)). With respect to the metric (34)

$$(D'_t x, y) = \mathrm{Sp}(t^*x + xt)y = \mathrm{Sp}(yt^* + ty)x = (D_t y, x).$$

Thus, the transformation $D'_t$ is adjoint to $D_t$ with respect to the metric (34). Therefore $\mathfrak{I}(\mathfrak{A}')$ consists of the transformations that are adjoint, with respect to the metric (34), to the transformations of $\mathfrak{I}(\mathfrak{A})$, and this means that it acts transitively in the cone $(V(\mathfrak{A}))'$ adjoint to $V(\mathfrak{A})$ with respect to the metric (34) (Proposition 9 of Chapter I). Thus,

$$V(\mathfrak{A}') = (V(\mathfrak{A}))'. \tag{39}$$

It now follows from (38) that

$$(V(\mathfrak{A}))' = \{t^*t:\ t \in \mathscr{T}(\mathfrak{A})\}. \tag{40}$$

## §7. The nilpotent part of a $T$-algebra

The scalar product

$$(a,\ b) = \operatorname{Sp} ab^* \tag{41}$$

introduces the structure of a Euclidean space into the $T$-algebra $\mathfrak{A}$, since, by axiom (V), the quadratic form $(a,\ a) = \operatorname{Sp} aa^*$ is positive-definite. The subspaces $\mathfrak{A}_{ij}$ are orthogonal with respect to the metric (41).

We consider the graded subalgebra

$$\mathfrak{N} = \sum_{i<j} \mathfrak{A}_{ij} \tag{42}$$

of $\mathfrak{A}$.

Definition 6. The graded algebra $\mathfrak{N}$ with the Euclidean metric (41) is called the *nilpotent part of* $\mathfrak{A}$.

The axioms for a $T$-algebra allow us to recover it from its nilpotent part. For, since

$$\dim \mathfrak{A}_{ij} = \dim \mathfrak{A}_{ji}, \qquad \dim \mathfrak{A}_{ii} = 1, \tag{43}$$

it follows that $\mathfrak{A}$ is uniquely recovered as a graded linear space. The involution $^*$ is also uniquely recovered, since we know that it is the identity on the subspaces $\mathfrak{A}_{ii}$ and maps $\mathfrak{A}_{ij}$ $(i < j)$ isomorphically onto $\mathfrak{A}_{ji}$.

Further, we can choose nonzero vectors $e_i$ arbitrarily in the one-dimensional subspaces $\mathfrak{A}_{ii}$ and make them idempotents. This uniquely defines the multiplication of diagonal matrices and also, by virtue of axiom (II), the multiplication of diagonal matrices by any others.

The space $\mathfrak{A}$ splits into a direct sum:

$$\mathfrak{A} = \mathfrak{N}^* + \mathfrak{H} + \mathfrak{N}, \tag{44}$$

where $\mathfrak{H}$ is the subspace of diagonal matrices. We know already how to multiply matrices in $\mathfrak{N}$ by one another and also matrices in $\mathfrak{H}$ by any matrices in $\mathfrak{N}$. The relation

$$a^* b^* = (ba)^* \quad (a,\ b \in \mathfrak{N}) \tag{45}$$

defines the operation of multiplication in $\mathfrak{N}^*$. Thus, it remains to "learn" how to multiply matrices in $\mathfrak{N}$ by matrices in $\mathfrak{N}^*$, in both orders. Before doing this we extend the metric (41) to the whole space $\mathfrak{A}$, using the facts that the subspaces $\mathfrak{N}^*$, $\mathfrak{H}$ and $\mathfrak{N}$ are orthogonal, that $(e_i,\ e_i) = n_i$, and that

$$(a^*,\ b^*) = \operatorname{Sp} a^* b = \operatorname{Sp} ba^* = (a,\ b). \tag{46}$$

For any $a,\ b \in \mathfrak{N}$

$$\operatorname{pr}_{\mathfrak{H}} ab^* = \sum a_{ij} b^*_{ij} = \sum \frac{(a_{ij},\ b_{ij})}{n_i}\, e_i. \tag{47}$$

In the same way

$$\operatorname{pr}_{\mathfrak{H}} a^* b = \sum a^*_{ij} b_{ij} = \sum \frac{(a_{ij},\ b_{ij})}{n_j}\, e_j. \tag{48}$$

These relations define the diagonal part of the matrices $ab^*$ and $a^*b$. Further, by axioms (III) and (IV), the following relations hold for any $c \in \mathfrak{N}$:

$$(ab^*, c^*) = \mathrm{Sp}\,(ab^*)\,c = \mathrm{Sp}\,(ca)\,b^* = (ca, b), \tag{49}$$

$$(a^*b, c^*) = \mathrm{Sp}\,(a^*b)\,c = \mathrm{Sp}\,(bc)\,a^* = (bc, a). \tag{50}$$

This defines the projections of $ab^*$ and $a^*b$ onto $\mathfrak{N}^*$. Finally, for any $c \in \mathfrak{N}$ we have

$$(ab^*, c) = (ba^*, c^*) = (cb, a), \tag{51}$$

$$(a^*b, c) = (b^*a, c^*) = (ac, b), \tag{52}$$

which defines the projections of $ab^*$ and $a^*b$ onto $\mathfrak{N}$. So the operation of multiplication in $\mathfrak{A}$ is completely recovered.

If we omit the middle terms in (46)–(52) and alter the formulation accordingly, then the above argument can be regarded as the construction of a graded algebra $\mathfrak{A}$ with an involution from the graded algebra $\mathfrak{N}$ equipped with the Euclidean metric. However, there is no guarantee in this case that $\mathfrak{A}$ will be a $T$-algebra. Let us consider the limitations that the axioms of a $T$-algebra impose on the original algebra $\mathfrak{N}$.

The axioms (I) and (II) of a $T$-algebra are satisfied by construction.

Let us verify axiom (III). For any $a, b \in \mathfrak{A}$ we have, by virtue of (47) and (48), that

$$\mathrm{pr}_{\mathfrak{H}}\,[ab] \triangleq \sum_{i<j} (a_{ij}b_{ji} - b_{ij}a_{ji}) + \sum_{i<j} (a_{ji}b_{ij} - b_{ji}a_{ij})$$
$$= \sum_{i<j} \frac{(a_{ij}, b_{ij}^*) - (b_{ij}, a_{ij}^*)}{n_i}\, e_i + \sum_{i<j} \frac{(a_{ji}^*, b_{ij}) - (b_{ji}^*, a_{ij})}{n_j}\, e_j,$$

so that

$$\mathrm{Sp}\,[ab] = \sum_{i<j} [(a_{ij}, b_{ji}^*) - (b_{ij}, a_{ji}^*) + (a_{ji}^*, b_{ij}) - (b_{ji}^*, a_{ij})] = 0.$$

By our construction of the Euclidean metric in $\mathfrak{A}$ it coincides with the metric defined by (41). In fact, for any $a, b \in \mathfrak{A}$

$$\mathrm{pr}_{\mathfrak{H}}\, ab^* = \sum_{i<j} a_{ij}b_{ij}^* + \sum a_{ii}b_{ii} + \sum_{i<j} a_{ji}b_{ji}^*$$
$$= \sum_{i<j} \frac{(a_{ij}, b_{ij})}{n_i}\, e_i + \sum \varrho\,(a_{ii})\,\varrho\,(b_{ii})\, e_i + \sum_{i<j} \frac{(a_{ji}^*, b_{ji}^*)}{n_j}\, e_j,$$

and

$$\operatorname{Sp} ab^* = \sum_{i<j} (a_{ij}, b_{ij}) + \sum n_i \varrho (a_{ii}) \varrho (b_{ii}) + \sum_{i<j} (a^*_{ji}, b^*_{ji})$$
$$= \sum_{i<j} (a_{ij}, b_{ij}) + \sum (a_{ii}, b_{ii}) + \sum_{i<j} (a_{ji}, b_{ji}) = (a, b).$$

The relation (45) is satisfied by construction for $a, b \in \mathfrak{N}$. It is not difficult to show that, in fact, it is satisfied for all $a, b$.

We now verify axiom (IV). If at least one of the matrices $a, b, c$ is diagonal, then $[abc] = 0$ and moreover $\operatorname{Sp}[abc] = 0$. Therefore we may assume that each of the matrices $a, b, c$ is contained either in $\mathfrak{N}$ or in $\mathfrak{N}^*$. Since

$$[x^*y^*z^*] = -[zyx]^*$$

for any $x, y, z \in \mathfrak{N}$, it suffices to consider the cases in which not more than one of the matrices $a, b, c$ lies in $\mathfrak{N}^*$. If $a, b, c \in \mathfrak{N}$, then $[abc] \in \mathfrak{N}$ and $\operatorname{Sp}[abc] = 0$. If $a$ or $c$ belongs to $\mathfrak{N}^*$ and the other two matrices belong to $\mathfrak{N}$, then axiom (IV) is a consequence of (49) or (50). Suppose finally that $a, c \in \mathfrak{N}$, $b \in \mathfrak{N}^*$. Using axioms (III) and (IV) in the cases in which they have already been proved we find that

$$\operatorname{Sp} a(bc) = \operatorname{Sp}(bc)a = \operatorname{Sp} b(ca) = \operatorname{Sp}(ca)b = \operatorname{Sp} c(ab) = \operatorname{Sp}(ab)c.$$

Axiom (V) is also satisfied in $\mathfrak{A}$, since

$$\operatorname{Sp} aa^* = (a, a) > 0,$$

if $a \neq 0$.

Axiom (VI) is satisfied if and only if the algebra $\mathfrak{N}$ is associative.

Axiom (VII) imposes the following limitations on $\mathfrak{N}$:

(A) for any $a_{ij}, c_{ij} \in \mathfrak{A}_{ij}$, $b_{jk} \in \mathfrak{A}_{jk}$ $(i < j < k)$

$$(a_{ij}b_{jk}, c_{ij}b_{jk}) = \frac{1}{n_j}(a_{ij}, c_{ij})(b_{jk}, b_{jk});$$

(B) if $a_{ik} \in \mathfrak{A}_{ik}$, $b_{jk} \in \mathfrak{A}_{jk}$ $(i < j < k)$ and $(a_{ik}, \mathfrak{N}b_{jk}) = 0$ then

$$(\mathfrak{N}a_{ik}, \mathfrak{N}b_{jk}) = 0.$$

In fact,

$$(a_{ij}b_{jk}, c_{ij}b_{jk}) = \operatorname{Sp}(a_{ij}b_{jk})(b^*_{jk}c^*_{ij}) = \operatorname{Sp}((a_{ij}b_{jk})b^*_{jk})c^*_{ij}$$
$$= \operatorname{Sp}(a_{ij}(b_{jk}b^*_{jk}))c^*_{ij} = \operatorname{Sp}(c^*_{ij}a_{ij})(b_{jk}b^*_{jk})$$
$$= n_j \varrho(c^*_{ij}a_{ij}) \varrho(b_{jk}b^*_{jk}) = \frac{1}{n_j}(a_{ij}, c_{ij})(b_{jk}, b_{jk}).$$

Further, if $a_{ik} \in \mathfrak{A}_{ik}$, $b_{jk} \in \mathfrak{A}_{jk}$ $(i < j < k)$, then for any $x, y \in \mathfrak{N}$

$$(xa_{ik}, yb_{jk}) = \sum_{s<i} (x_{si}a_{ik}, y_{sj}b_{jk}) = \sum_{s<i} \operatorname{Sp}(x_{si}a_{ik})(b^*_{jk}y^*_{sj}) =$$

$$= \sum_{s<i} \mathrm{Sp}\,[((x_{si}a_{ik})\,b^*_{jk})\,y^*_{sj} + ((x_{si}b_{jk})\,a^*_{ik})\,y^*_{sj}]$$

$$= \sum_{s<i} \mathrm{Sp}\,(x_{si}\,(a_{ik}b^*_{jk} + b_{jk}a^*_{ik}))\,y^*_{sj} = \sum_{s<i} \mathrm{Sp}\,(x_{si}\,(a_{ik}b^*_{jk}))\,y^*_{sj}.$$

If $(a_{ik}, \mathfrak{R}b_{jk}) = 0$, then, for any $c_{ij}$

$$(a_{ik}b^*_{jk},\, c_{ij}) = (c_{ij}b_{jk},\, a_{ik}) = 0$$

and $a_{ik}b^*_{jk} = 0$. Therefore, in this case

$$(xa_{ik},\, yb_{jk}) = 0 \quad (x,\, y \in \mathfrak{R}).$$

We now show that if $\mathfrak{R}$ is associative and satisfies the conditions (A) and (B), then the algebra $\mathfrak{A}$ satisfies axiom (VII$'$). We may assume that

$$t = a_{ij} \in \mathfrak{A}_{ij}, \quad u = b_{jk} \in \mathfrak{A}_{jk}, \quad w = c_{sk} \in \mathfrak{A}_{sk}.$$

We must show that

$$[a_{ij}b_{jk}c^*_{sk}] + [a_{ij}c_{sk}b^*_{jk}] = 0. \tag{53}$$

We observe that the second term is 0, because $s \neq j$. We consider five cases separately.

1) $j = s$. In this case the identity (53) is equivalent to the fact that for any $a_{ij} \in \mathfrak{A}_{ij}$, $b_{jk} \in \mathfrak{A}_{jk}$,

$$[a_{ij}b_{jk}b^*_{jk}] = 0 \quad (i < j < k). \tag{54}$$

Clearly $[a_{ij}b_{jk}b^*_{jk}] \in \mathfrak{A}_{ij}$. Suppose that $x_{ij}$ is an arbitrary element of $\mathfrak{A}_{ij}$. Then, by virtue of (A)

$$((a_{ij}b_{jk})\,b^*_{jk},\, x_{ij}) = (a_{ij}b_{jk},\, x_{ij}b_{jk})$$
$$= \frac{1}{n_j}\,(a_{ij},\, x_{ij})\,(b_{jk},\, b_{jk}) = (a_{ij},\, x_{ij})\,\varrho\,(b_{jk}b^*_{jk}) = (a_{ij}\,(b_{jk}b^*_{jk}),\, x_{ij}).$$

Therefore $(a_{ij}b_{jk})b^*_{jk} = a_{ij}(b_{jk}b^*_{jk})$.

2) $j < s < k$. The space $\mathfrak{A}_{jk}$ splits into the direct sum

$$\mathfrak{A}_{jk} = \mathfrak{A}_{js}c_{sk} + \mathfrak{B},$$

where $\mathfrak{B}$ is the orthocomplement of $\mathfrak{A}_{js}c_{sk}$. If $b_{jk} \in \mathfrak{A}_{js}c_{sk}$, then $b_{jk} = u_{js}c_{sk}$ for some $u_{js} \in \mathfrak{A}_{js}$. It follows from the associativity of $\mathfrak{R}$ and (54) that

$$a_{ij}\,(b_{jk}c^*_{sk}) = a_{ij}\,((u_{js}c_{sk})\,c^*_{sk}) = a_{ij}\,(u_{js}\,(c_{sk}c^*_{sk}))$$
$$= (a_{ij}u_{js})\,(c_{sk}c^*_{sk}) = ((a_{ij}u_{js})\,c_{sk})\,c^*_{sk} = (a_{ij}\,(u_{js}c_{sk}))\,c^*_{sk} = (a_{ij}b_{jk})\,c^*_{sk}.$$

Now suppose that $b_{jk} \in \mathfrak{B}$. Then

$$(b_{jk},\, \mathfrak{R}c_{sk}) = 0$$

and, by virtue of (B),

$$(\mathfrak{R}b_{jk},\quad \mathfrak{R}c_{sk}) = 0.$$

For every $x \in \mathfrak{N}$

$$(b_{jk}c_{sk}^*, x) = (b_{jk}, xc_{sk}) = 0,$$

so that $b_{jk}c_{sk}^* = 0$ and

$$a_{ij}(b_{jk}c_{sk}^*) = 0.$$

On the other hand, for every $x \in \mathfrak{N}$

$$((a_{ij}b_{jk})c_{sk}^*, x) = (a_{ij}b_{jk}, xc_{sk}) = 0,$$

and therefore

$$(a_{ij}b_{jk})c_{sk}^* = 0.$$

3) $i < s < j$. Clearly $[a_{ij}b_{jk}c_{sk}^*] \in \mathfrak{A}_{is}$. Let $x_{is}$ be an arbitrary element in $\mathfrak{A}_{is}$. Using case 2), which we have already discussed, we find that

$$(a_{ij}(b_{jk}c_{sk}^*), x_{is}) = (a_{ij}(c_{sk}b_{jk}^*)^*, x_{is}) = (a_{ij}, (x_{is}c_{sk})b_{jk}^*) = \\ = (a_{ij}b_{jk}, x_{is}c_{sk}) = ((a_{ij}b_{jk})c_{sk}^*, x_{is}).$$

4) $s < i$. We have

$$[a_{ij}b_{jk}c_{sk}^*]^* = -[c_{sk}b_{jk}^*a_{ij}^*] \in \mathfrak{A}_{si}.$$

For any $x_{si}$

$$(c_{sk}(b_{jk}^*a_{ij}^*), x_{si}) = (c_{sk}(a_{ij}b_{jk})^*, x_{si}) = (c_{sk}, x_{si}(a_{ij}b_{jk})) = (c_{sk}, (x_{si}a_{ij})b_{jk}) \\ = (c_{sk}b_{jk}^*, x_{si}a_{ij}) = ((c_{sk}b_{jk}^*)a_{ij}^*, x_{si}).$$

5) $s = i$. In this case

$$[a_{ij}b_{jk}c_{sk}^*] = \varrho([a_{ij}b_{jk}c_{ik}^*])e_i = \frac{e_i}{n_i}\operatorname{Sp}[a_{ij}b_{jk}c_{ik}^*] = 0.$$

Definition 7. An associative algebra $\mathfrak{N}$, graded by subspaces $\mathfrak{N}_{ij}$ $(i < j;$ $i, j = 1, \cdots, m)$ and equipped with a Euclidean metric is called an *N-algebra of rank m* if the following conditions are satisfied:

(I) $\mathfrak{N}_{ij}\mathfrak{N}_{jk} \subset \mathfrak{N}_{ik}$;

(II) $\mathfrak{N}_{ij}\mathfrak{N}_{lk} = 0$ if $j \neq l$;

(III) $(\mathfrak{N}_{ij}, \mathfrak{N}_{kl}) = 0$ if $i \neq k$ or $j \neq l$;

(IV) for any $a_{ij} \in \mathfrak{N}_{ij}$, $b_{jk} \in \mathfrak{N}_{jk}$

$$(a_{ij}b_{jk}, a_{ij}b_{jk}) = \frac{1}{n_j}(a_{ij}, a_{ij})(b_{jk}, b_{jk}),$$

where

$$n_j = 1 + \frac{1}{2}\sum_{s<j}\dim \mathfrak{N}_{sj} + \frac{1}{2}\sum_{s>j}\dim \mathfrak{N}_{js};\ ^{1)}$$

1) The $n_j$ may be taken to be arbitrary positive numbers (cf. the footnote on p. 380).

(V) if $a_{ik} \in \mathfrak{N}_{ik}$, $b_{jk} \in \mathfrak{N}_{jk}$ $(i < j)$ and $(a_{ik}, \mathfrak{N}b_{jk}) = 0$,

then

$$(\mathfrak{N}a_{ik}, \mathfrak{N}b_{jk}) = 0.$$

Axiom (IV) for an $N$-algebra is equivalent to condition (A), which is obtained from axiom (IV) by polarization with respect to the $a_{ij}$. Therefore, the nilpotent part of any $T$-algebra is an $N$-algebra. For isomorphic $T$-algebras to correspond to isomorphic $N$-algebras we must give the following definition of isomorphic $N$-algebras.

Definition 8. A mapping $C$ of an $N$-algebra $\mathfrak{N}$ of rank $m$ onto an $N$-algebra $\widetilde{\mathfrak{N}}$ of the same rank is said to be *isomorphic* if it is an algebra isomorphism, preserves the Euclidean metric, and transforms each subspace $\mathfrak{N}_{ij} \subset \mathfrak{N}$ into a subspace $\widetilde{\mathfrak{N}}_{\widetilde{i}\widetilde{j}} \subset \widetilde{\mathfrak{N}}$, where $i \to \widetilde{i}$ is a permutation of the set $\{1, \cdots, m\}$ such that $i < j$, $\widetilde{i} > \widetilde{j}$ imply that $\mathfrak{N}_{ij} = \widetilde{\mathfrak{N}}_{\widetilde{i}\widetilde{j}} = 0$. Two $N$-algebras are said to be *isomorphic* if there exists an isomorphic mapping of one onto the other.

Summing up the results we have obtained we can formulate

Proposition 3. *The mapping that associates with every $T$-algebra its nilpotent part induces a one-to-one mapping of the set of all classes of isomorphic $T$-algebras onto the set of all classes of isomorphic $N$-algebras.*

## §8. Convex homogeneous cones of rank 3

$T$-algebras of rank 3 are the first examples of nonselfadjoint convex homogeneous cones. Also they are typical examples: using them we can examine many of the properties of arbitrary convex homogeneous cones.

By the results of §7, a $T$-algebra $\mathfrak{A}$ of rank 3 is completely determined by the numbers $n_{12}, n_{23}, n_{13}$ (cf. the notation in §1), by the Euclidean spaces $\mathfrak{A}_{12}, \mathfrak{A}_{23}, \mathfrak{A}_{13}$ of corresponding dimensions and by the bilinear mapping $(a, b) \to ab$ of the product $\mathfrak{A}_{12} \times \mathfrak{A}_{23}$ into $\mathfrak{A}_{13}$, satisfying the condition

$$(ab, ab) = \kappa(a, a)(b, b),$$

where $\kappa$ is a positive factor of no essential significance. The last condition implies, in particular, that if $n_{12} \neq 0$ and $n_{23} \neq 0$, then $n_{13} \geq \max\{n_{12}, n_{23}\}$. The equations of the corresponding cone can be written down without any difficulty (cf. §3) but, as a rule they do not supply any interesting information. The adjoint cone is obtained when we interchange the roles of $\mathfrak{A}_{12}$ and $\mathfrak{A}_{23}$ (cf. §6).

If $n_{12} = n_{23} = n_{13} = \nu$ we can only have the values $\nu = 0, 1, 2, 4, 8$, and once $\nu$ is given, the $T$-algebra $\mathfrak{A}$ is uniquely determined. All the corresponding cones are *selfadjoint*.

If $n_{12} = 0$, then the numbers $n_{23}$ and $n_{13}$ can be arbitrary and they completely determine the $T$-algebra $\mathfrak{A}$. If in this case $n_{23}, n_{13} \neq 0$, then, as is easy to see, the cone $V(\mathfrak{A})$ is not selfadjoint (and is not even isomorphic to its adjoint). In particular, when $n_{23} = n_{13} = 1$, we obtain the *simplest example of a nonselfadjoint convex homogeneous cone.* Its dimension is 5. It is isomorphic to the cone consisting of all pairs

$$\left\{\begin{pmatrix} a & x \\ x & c \end{pmatrix}, \quad \begin{pmatrix} b & y \\ y & c \end{pmatrix}\right\}$$

of positive-definite, symmetric, real, 2nd-order matrices with a common corner element $c$. The adjoint cone can be described as the set of positive-definite symmetric matrices of the form

$$\begin{pmatrix} a & y & x \\ y & b & 0 \\ x & 0 & c \end{pmatrix}.$$

If $n_{12} = n_{23} = 2$, $n_{13} = 4$, then we obtain a one-parameter family of nonisomorphic $T$-algebras. It corresponds to a *one-parameter family of nonisomorphic convex homogeneous cones of dimension* 11. It is not difficult to verify that for lower dimensions there is only a finite number of nonisomorphic convex homogeneous cones (of any rank).

## §9. Proof of universality

Theorem 4. *For every convex homogeneous cone* V *there exists a unique* (*up to isomorphism*) *$T$-algebra $\mathfrak{A}$ such that the cones $V$ and $V(\mathfrak{A})$ are isomorphic.*

Proof. The algebra of the convex cone $V(\mathfrak{A})$ is a clan $\mathfrak{L}(\mathfrak{A})$ (cf. §2) which is given in $\mathfrak{X}$ by the operation $\Delta$ (cf. formula (16)). By Theorem 2 the problem reduces to proving that for every clan $\mathfrak{L}$ with a unit element there exists a unique $T$-algebra $\mathfrak{A}$ such that $\mathfrak{L} \simeq \mathfrak{L}(\mathfrak{A})$.

Let $\mathfrak{L}$ be a clan with a unit element and

$$\mathfrak{L} = \sum_{i \leqslant j} \mathfrak{L}_{ij}$$

be its normal decomposition (cf. §4 of Chapter II). For every space $\mathfrak{L}_{ij}$ with $i < j$ we take the linear space $\mathfrak{A}_{ij}$ of the same dimension and we fix an isomorphic mapping

$$x \longrightarrow \hat{x} \in \mathfrak{A}_{ij}.$$

of $\mathfrak{L}_{ij}$ onto $\mathfrak{A}_{ij}$. We form the direct sum

$$\mathfrak{N} = \sum_{i<j} \mathfrak{A}_{ij} \tag{55}$$

of the linear spaces $\mathfrak{A}_{ij}$ and for every

$$x = \sum_{i<j} x_{ij} \in \mathfrak{L} \qquad (x_{ij} \in \mathfrak{L}_{ij})$$

we put

$$\hat{x} = \sum_{i<j} \hat{x}_{ij} \in \mathfrak{N}. \tag{56}$$

We shall assume that the algebra $\mathfrak{L}$ is equipped with the canonical Euclidean metric

$$(x, y) = \operatorname{Sp} L_{x \triangle y}. \tag{57}$$

We equip $\mathfrak{N}$ with a Euclidean metric by putting

$$(\hat{x}, \hat{y}) = \frac{1}{2}(x, y) \tag{58}$$

for any $x, y \in \mathfrak{L}$. We define an operation of multiplication in $\mathfrak{N}$ by the formula

$$\hat{x}\hat{y} = \sum_{i<j<k} \widehat{x_{ij} \triangle y_{jk}} \qquad (x, y \in \mathfrak{L}). \tag{59}$$

It follows from (43) of Chapter II that the multiplication defined in this way is associative.

We shall show that $\mathfrak{N}$ equipped with the grading (55), with the Euclidean metric (58), and with the associative multiplication (59) is an $N$-algebra. The axioms (I) – (III) for an $N$-algebra are obviously satisfied. By (42) of Chapter II, for all $x_{ij} \in \mathfrak{L}_{ij}$, $y_{jk} \in \mathfrak{L}_{jk}$ $(i < j < k)$ we have

$$(\hat{x}_{ij}\hat{y}_{jk}, \hat{x}_{ij}\hat{y}_{jk}) = (\widehat{x_{ij} \triangle y_{jk}}, \widehat{x_{ij} \triangle y_{jk}}) = \frac{1}{2}(x_{ij} \triangle y_{jk}, x_{ij} \triangle y_{jk})$$
$$= \frac{1}{4s(e_j)}(x_{ij}, x_{ij})(y_{jk}, y_{jk}) = \frac{1}{s(e_j)}(\hat{x}_{ij}, \hat{x}_{ij})(\hat{y}_{jk}, \hat{y}_{jk}),$$

where $e_j \in \mathfrak{L}_{jj}$ is an idempotent of $\mathfrak{L}$ and $s(e_j) = \operatorname{Sp} L_{e_j}$. It is not difficult to see from table (28) of Chapter II that

$$\operatorname{Sp} L_{e_j} = 1 + \frac{1}{2}\left(\sum_{s<j} \dim \mathfrak{L}_{sj} + \sum_{s>j} \dim \mathfrak{L}_{js}\right).$$

Thus,

$$(\hat{x}_{ij}\hat{y}_{jk}, \hat{x}_{ij}\hat{y}_{jk}) = \frac{1}{n_j}(\hat{x}_{ij}, \hat{x}_{ij})(\hat{y}_{jk}, \hat{y}_{jk}),$$

where

$$n_j = 1 + \frac{1}{2}\left(\sum_{s<j} \dim \mathfrak{A}_{sj} + \sum_{s>j} \dim \mathfrak{A}_{js}\right),$$

i.e., $\mathfrak{N}$ satisfies the axiom (IV) for a $T$-algebra. We finally verify axiom (V). Let $x_{ik} \in \mathfrak{L}_{ik}$, $y_{jk} \in \mathfrak{L}_{jk}$ $(i < j < k)$ be such that

$$(\hat{x}_{ik}, \ \mathfrak{N}\hat{y}_{jk}) = 0.$$

Then

$$(x_{ik}, \ \mathfrak{L}_{ij} \triangle y_{jk}) = 0,$$

and the relation (41) of Chapter II implies that

$$x_{ik} \triangle y_{jk} = 0.$$

For any $z_{si} \in \mathfrak{L}_{si}$, $u_{sj} \in \mathfrak{L}_{sj}$ $(s < i)$

$$(\hat{z}_{si}\hat{x}_{ik}, \ \hat{u}_{sj}\hat{y}_{jk}) = \frac{1}{2}(z_{si} \triangle x_{ik}, \ u_{sj} \triangle y_{jk}).$$

Using (41) and (44) of Chapter II we find that

$$(z_{si} \triangle x_{ik}, \ u_{sj} \triangle y_{jk}) = ((z_{si} \triangle x_{ik}) \triangle y_{jk}, \ u_{sj}) = (z_{si} \triangle (x_{ik} \triangle y_{jk}), \ u_{sj}) = 0.$$

Therefore

$$(\mathfrak{N}\hat{x}_{ik}, \ \mathfrak{N}\hat{y}_{jk}) = 0.$$

Thus, $\mathfrak{N}$ is an $N$-algebra. We construct (cf. §7) the $T$-algebra

$$\mathfrak{A} = \sum \mathfrak{A}_{ij},$$

of which $\mathfrak{N}$ is the nilpotent part. We shall prove that $\mathfrak{L} \simeq \mathfrak{L}(\mathfrak{A})$.

Let $\mathfrak{X}$ be the space of Hermitian matrices of $\mathfrak{A}$. With every element

$$x = \sum \lambda_i e_i + \sum_{i<j} x_{ij} \qquad (x_{ij} \in \mathfrak{L}_{ij})$$

of $\mathfrak{L}$ we associate the Hermitian matrix

$$\sigma(x) = \sum \lambda_i e_i + \sum_{i<j} \hat{x}_{ij} + \sum_{i<j} (\hat{x}_{ij})^* \in \mathfrak{X}.$$

(We observe that the idempotents $e_i \in \mathfrak{L}_{ii}$ of $\mathfrak{L}$ and the idempotents $e_i \in \mathfrak{A}_{ii}$ of $\mathfrak{A}$ are denoted by the same symbols. This will not lead to confusion, especially since $\sigma(e_i) = e_i$.)

It is immediately verified (cf. formulae (34) – (37), (40) – (44) of Chapter II and (49) – (52) of the present chapter) that the mapping $\sigma$ is an isomorphism of $\mathfrak{L}$ onto $\mathfrak{L}(\mathfrak{A})$.

We now show that if the clans $\mathfrak{L}(\mathfrak{A}^{(p)})$ $(p = 1, 2)$ are isomorphic, then so are the $T$-algebras $\mathfrak{A}^{(p)}$. Let $\mathfrak{X}^{(p)}$ be the space of Hermitian matrices of $\mathfrak{A}^{(p)}$. We put

$$\mathfrak{X}_{ij}^{(p)} = \mathfrak{X}^{(p)} \cap (\mathfrak{A}_{ij} + \mathfrak{A}_{ji}).$$

The decomposition

$$\mathfrak{L}(\mathfrak{A}^{(p)}) = \mathfrak{X}^{(p)} = \sum_{i \leqslant j} \mathfrak{X}_{ij}^{(p)} \qquad (60)$$

is a normal decomposition of the clan $\mathfrak{L}(\mathfrak{A}^{(p)})$ (cf. Definition 6 of Chapter II). We take an isomorphic mapping $C$ of $\mathfrak{L}(\mathfrak{A}^{(1)})$ onto $\mathfrak{L}(\mathfrak{A}^{(2)})$. Proposition 10 of Chapter II allows us to claim that there exists a permutation $i \to \tilde{i}$ such that

$$C\mathfrak{X}_{ij}^{(1)} = \mathfrak{X}_{\tilde{i}\tilde{j}}^{(2)}$$

and $i < j$, $\tilde{i} > \tilde{j}$ imply that $\mathfrak{X}_{ij}^{(1)} = 0$. Let $\mathfrak{N}^{(p)}$ denote the nilpotent part of $\mathfrak{A}^{(p)}$ and for every $x \in \sum_{i<j} \mathfrak{X}_{ij}^{(1)}$ put

$$\hat{C}\hat{x} = \widehat{Cx} \tag{61}$$

(cf. (11)). It is easy to verify that the mapping $\hat{C}$ is an isomorphism of $\mathfrak{N}^{(1)}$ onto $\mathfrak{N}^{(2)}$.

## §10. Convex homogeneous domains

The apparatus of $T$-algebras allows us to describe not only the convex homogeneous cones, but also arbitrary convex homogeneous domains.

Let $U$ be a convex homogeneous domain, $V(U)$ the cone fitted onto it and $\mathfrak{A}$ the (uniquely determined) $T$-algebra such that $V(U) \simeq V(\mathfrak{A})$. The domain $U$ is then realized as the cross-section of $V(\mathfrak{A})$ by a hyperplane $P$ of the space $\mathfrak{X}$ of Hermitian matrices of $\mathfrak{A}$. There exists a maximal connected triangular subgroup $\mathfrak{T}$ of $\mathfrak{G}(V(\mathfrak{A}))$ that splits into the direct product

$$\mathcal{T} = \mathcal{T}_1 \times \{\lambda E\}_{\lambda>0},$$

where $\mathfrak{T}_1$ leaves $P$ invariant and acts transitively in $U$. Since the maximal connected triangular subgroups are conjugate in $\mathfrak{G}(V(\mathfrak{A}))$, we may assume that $\mathfrak{T} = \mathfrak{T}(\mathfrak{A})$. We may also assume that $P$ passes through the point $e \in V(\mathfrak{A})$. Then

$$P = \mathfrak{X}_1 + e,$$

where $\mathfrak{X}_1$ is a subspace of codimension 1 and $\mathfrak{X}$ is invariant with respect to $\mathfrak{T}(\mathfrak{A})$. The infinitesimal transformations $D_t$, $t \in \mathfrak{T}$ (cf. formula (24)), generate $\mathfrak{T}(\mathfrak{A})$ and leave $\mathfrak{X}_1$ invariant. Conversely, if $\mathfrak{X}_1 \subset \mathfrak{X}$ is a subspace of codimension 1 that is invariant with respect to the transformations $D_t$, $t \in \mathfrak{T}$, and, hence, with respect to $\mathfrak{T}(\mathfrak{A})$, then the convex domain $(\mathfrak{X}_1 + e) \cap V(\mathfrak{A})$ is homogeneous (Proposition 11 of Chapter I).

**Proposition 4.** *If* $n_{i_0 s} = 0$ *in* $\mathfrak{A}$ *for all* $s > i_0$, *then the convex domain*

$$U = \{x \in V(\mathfrak{A}) : \; x_{i_0 i_0} = 1\} \tag{62}$$

*is homogeneous. All convex homogeneous domains can be obtained in this way.*

Proof. Suppose that $n_{i_0 s} = 0$ in $\mathfrak{A}$ for all $s > t_0$. Then the space

$$\mathfrak{X}_1 = \{x \in \mathfrak{X} : \; x_{i_0 i_0} = 0\} \subset \mathfrak{X} \tag{63}$$

is invariant with respect to the transformations $D_t$, $t \in \mathfrak{T}$; in fact, for any $x \in \mathfrak{X}_1$ and for any $t \in \mathfrak{T}$

$$(D_t x)_{i_0 i_0} = (tx + xt^*)_{i_0 i_0} = \sum_{s \geqslant i_0} (t_{i_0 s} x_{s i_0} + x_{i_0 s} t^*_{i_0 s}) = 0.$$

Therefore the convex domain

$$(\mathfrak{X}_1 + e) \cap V(\mathfrak{A}) = \{x \in V(\mathfrak{A}) : \; x_{i_0 i_0} = 1\}$$

is homogeneous.

Now suppose that $\mathfrak{A}$ is an arbitrary $T$-algebra and that $\mathfrak{X}_1$ is an invariant subspace of codimension 1 in the space $\mathfrak{X}$ of Hermitian matrices of the algebra $\mathfrak{A}$. If

$$x = \sum x_{ii} + \sum_{i<j} (x_{ij} + x_{ji}) \in \mathfrak{X}_1,$$

then the Hermitian matrices

$$x_{ii} = \frac{1}{2}(D^2_{e_i} - D_{e_i}) x$$

and

$$x_{ij} + x_{ji} = D_{e_i} D_{e_j} x \qquad (i < j)$$

are also contained in $\mathfrak{X}_1$. Therefore $\mathfrak{X}_1$ is the sum of its intersections with the subspaces $\mathfrak{X}_{ij} = \mathfrak{X} \cap (\mathfrak{A}_{ij} + \mathfrak{A}_{ji})$. If $e_i \in \mathfrak{X}_1$, then

$$\sum_{p \leqslant i} \mathfrak{X}_{pi} \subset \mathfrak{X}_1,$$

since, for any $a_{pi} \in \mathfrak{A}_{pi}$ $(p \leq i)$,

$$a_{pi} + a^*_{pi} = D_{a_{pi}} e_i.$$

Therefore there exists an $i_0$ such that $e_{i_0} \notin \mathfrak{X}_1$. Clearly, then

$$\mathfrak{X}_1 = \{x \in \mathfrak{X} : \; x_{i_0 i_0} = 0\}.$$

For any $s > i_0$ and $a_{i_0 s} \in \mathfrak{A}_{i_0 s}$ we have

$$D_{a_{i_0 s}} (a_{i_0 s} + a^*_{i_0 s}) = 2 a_{i_0 s} a^*_{i_0 s} = \frac{2(a_{i_0 s}, a_{i_0 s})}{n_{i_0}} e_{i_0} \in \mathfrak{X}_1,$$

whence $(a_{i_0 s}, a_{i_0 s}) = 0$ and $a_{i_0 s} = 0$. Therefore $n_{i_0 s} = 0$ for all $s > i_0$.

Thus, all the invariant subspaces $\mathfrak{X}_1 \subset \mathfrak{X}$ of codimension 1 are of the form (63). In view of the remark made at the beginning of this section, this implies that any convex homogeneous domain $U$ for which $V(U) \simeq V(\mathfrak{A})$ is isomorphic to a domain of the type (62).

We observe that the indices $i_0$ satisfying the condition of Proposition 4 can be characterized by the fact that by means of an inessential change in the grading of $\mathfrak{A}$ the idempotent $e_{i_0}$ can be transferred to the last place.

## BIBLIOGRAPHY


[1] N. Bourbaki, *Espaces vectoriels topologiques*, Livre V, Éléments de Mathématique, Actualités Sci. Ind. Nos. 1189, 1229, 1230, Hermann, Paris, 1953, 1955; Russian transl., IL, Moscow, 1959. MR 14, 880; MR 17, 1109.

[2] B. A. Venkov and B. B. Venkov, *On the automorphisms of a convex cone,* Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. 17 (1962), no. 7, 42–57. (Russian. English summary) MR 28 #5377.

[3] È. B. Vinberg, *Homogeneous cones*, Dokl. Akad. Nauk SSSR 133 (1960), 9–12 = Soviet Math. Dokl. 1 (1960), 787–790. MR 25 #5077.

[4] ———, *The Morozov-Borel theorem for real Lie groups*, Dokl. Akad. Nauk SSSR 141 (1961), 270–273 = Soviet Math. Dokl. 2 (1961), 1416–1419. MR 26 #252.

[5] ———, *Convex homogeneous domains*, Dokl. Akad. Nauk SSSR 141 (1961), 521–524 = Soviet Math. Dokl. 2 (1961), 1470–1473. MR 26 #253.

[6] ———, *Automorphisms of homogeneous convex cones*, Dokl. Akad. Nauk SSSR 143 (1962), 265–268 = Soviet Math. Dokl. 3 (1962), 371–374. MR 24 #2604.

[7] È. B. Vinberg, S. G. Gindikin and I. I. Pjateckiĭ-Šapiro, *Classification and canonical realization of complex bounded homogeneous domains*, Trudy Moskov. Mat. Obšč. 12 (1963), 359–388. (Russian) MR 28 #1638.

[8] S. G. Gindikin, *Integral formulas for Siegel domains of second kind*, Dokl. Akad. Nauk SSSR 141 (1961), 531–534 = Soviet Math. Dokl. 2 (1961), 1480–1483. MR 24 #A2053.

[9] E. Cartan, *Domaines bornés homogènes de l'espace de n variables complexes*, Abh. Math. Sem. Hamburg Univ. 11 (1935), 116–162.

[10] M. Köcher, *Positivitätsbereiche im* $R^n$, Amer. J. Math. 79 (1957), 575–596.

[11] ———, *Analysis in reellen Jordan Algebren*, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. IIa 1958, 67–74.

[12] ———, *Die Geodätischen von Positivitätsbereichen*, Math. Ann. 135 (1958), 192–202.

[13] J.-L. Koszul, *Domaines bornés homogènes et orbites de groupes de transformations affines*, Bull. Soc. Math. France 89 (1961), 515–533. MR 26 #3090.

[14] ———, *Ouverts convexes homogènes des espaces affines*, Math. Z. 79 (1962), 254–259. MR 27 #774.

[15] I. I. Pjateckiĭ-Šapiro, *Estimate of the dimensionability of the space of automorphous forms for certain types of discrete groups*, Dokl. Akad. Nauk SSSR 113 (1957), 980–983. (Russian) MR 20 #5888.

[16] ———, *Some questions in harmonic analysis in homogeneous cones*, Dokl. Akad. Nauk SSSR 116 (1957), 181–184. (Russian) MR 23 #A475.

[17] ———, *On a problem proposed by E. Cartan*, Dokl. Akad. Nauk SSSR 124 (1959), 272–273. (Russian) MR 21 #728.

[18] ———, *Geometry of classical domains and theory of automorphic functions*, Sovromennye Problemy Matematiki, Fizmatgiz, Moscow, 1961. (Russian) MR 25 #231.

[19] ———, *Classification of bounded homogeneous regions in n-dimensional complex space*, Dokl. Akad. Nauk SSSR 141 (1961), 316–319 = Soviet Math. Dokl. 2 (1961), 1460–1463. MR 24 #A2681.

[20] ———, *On bounded homogeneous domains in n-dimensional complex space*, Izv. Akad. Nauk SSSR Ser. Mat. 26 (1962), 107–124; English transl., Amer. Math. Soc. Transl. (2) 43 (to appear).

[21] ———, *The structure of j-algebras*, Izv. Akad. Nauk SSSR Ser. Mat. 26 (1962), 453–484. (Russian) MR 25 #4553.

[22] O. S. Rothaus, *Domains of positivity*, Abh. Math. Sem. Univ. Hamburg 24 (1960), 189–235. MR 22 #12540.

[23] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge Univ. Press, New York, 1934; 2nd ed., 1952; Russian transl. of 2nd ed., IL, Moscow, 1948. MR 13, 727; MR 18, 722.

[24] C. Hertneck, *Positivitätsbereiche und Jordan-Strukturen*, Math. Ann. 146 (1962), 433–455. MR 26 #1489.

Translated by:

J. Burlak