Tight bounds on Lyapunov rank

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Abstract

The Lyapunov rank of a cone is the number of independent equations obtainable from an analogue of the complementary slackness condition in cone programming problems, and more equations are generally thought to be better. Bounding the Lyapunov rank of a proper cone in $\mathbb{R}^n$ from above is an open problem. Gowda and Tao gave an upper bound of $n^2 - n$ that was later improved by Orlitzky and Gowda to $(n-1)^2$. We settle the matter and show that the Lyapunov rank of $\left(n^2 - n\right)/2 + 1$ belonging to the Lorentz second-order cone is maximal.

1 Preliminaries

The study of Lyapunov rank was initiated by Rudolf, Noyan, Papp, and Alizadeh [9], who called it *bilinearity rank*. The idea is inspired by the complementary slackness condition in linear programming over the nonnegative orthant $\mathbb{R}^n_+$. When the entries of x, s are nonnegative, the single complementarity condition $\langle x, s \rangle = 0$ can be split into n equations $x_i s_i = 0$ for $i = 1, 2, \ldots, n$. If $\{e_1, e_2, \ldots, e_n\}$ denotes the standard basis and if we define $L_i \coloneqq e_i e_i^T$, then we have obtained the new equations $\langle L_i x, s \rangle = x_i s_i = 0$ from $\langle I x, s \rangle = 0$. The matrices $\{L_i \mid i = 1, 2, \ldots, n\}$ form a basis for a vector space of dimension n whose elements L all satisfy the same property: if $x, s \in \mathbb{R}^n_+$ and if $\langle x, s \rangle = 0$, then $\langle L x, s \rangle = 0$. This is how we obtain n equations from $I = L_1 + L_2 + \cdots + L_n$.

This situation generalizes as follows. Suppose now that K is any closed convex cone in $\mathbb{R}^n$. The *dual cone* of K is

$$K^* \coloneqq \{y \in \mathbb{R}^n \mid \langle x, y \rangle \geq 0 \text{ for all } x \in K\}$$

and in order to solve optimization problems over K, we often seek pairs of $x \in K$ and $s \in K^*$ such that $\langle x, s \rangle = 0$. We call this set

$$C(K) \coloneqq \{(x, s) \in K \times K^* \mid \langle x, s \rangle = 0\}$$

the *complementarity set* of the cone K. Gowda [3] calls any matrix L preserving the complementarity of these pairs *Lyapunov-like*, after the Lyapunov operators in the theory of dynamical systems. The matrices Lyapunov-like on K form a vector subspace that we denote by

$$\mathbf{LL}(K) := \left\{ L \in \mathbb{R}^{n \times n} \;\middle|\; (x, s) \in C(K) \text{ implies } \langle Lx, s \rangle = 0 \right\},$$

and they are the same "bilinear complementarity relations" that Rudolf et alii studied in the context of conic optimization [9]. The dimension of this vector space is the *Lyapunov rank* [5] of K, and is denoted by

$$\beta(K) := \dim(\mathbf{LL}(K)).$$

Clearly, the identity matrix belongs to $\mathbf{LL}(K)$ regardless of K. The identity can therefore be expressed in terms of a basis $\{L_i \mid i = 1, 2, \ldots, \beta(K)\}$ for $\mathbf{LL}(K)$, whence we obtain the $\beta(K)$ equations $\langle L_i x, s \rangle = 0$ from $\langle Ix, s \rangle = 0$.

Lyapunov rank has been computed for a plethora of proper cones. Of paramount interest is the Lorentz second-order cone,

$$\mathcal{L}_+^n := \left\{ (t, x)^T \in \mathbb{R} \times \mathbb{R}^{n-1} \;\middle|\; \|x\| \le t \right\},$$

whose Lyapunov rank was shown to be $\left(n^2 - n\right)/2 + 1$ by Gowda and Tao [5]. Gowda [3] had uncovered a deep connection, recognizing that the matrices Lyapunov-like on a cone are the Lie algebra of its automorphism group. If

$$\mathrm{Aut}(K) := \left\{ A \in \mathbb{R}^{n \times n} \;\middle|\; A^{-1} \text{ exists and } A(K) = K \right\}$$

and if $\mathrm{Lie}(G)$ is the Lie algebra of the Lie group G, then we have

$$\begin{aligned} \mathbf{LL}(K) &= \mathrm{Lie}(\mathrm{Aut}(K)); \text{ thus} \\ \beta(K) &= \dim(\mathrm{Lie}(\mathrm{Aut}(K))). \end{aligned}$$

We adopt this definition of Lyapunov rank. No generality is lost by working in $\mathbb{R}^n$, since all finite-dimensional real inner-product spaces are isometric, and

Proposition 1 (Orlitzky [7], Proposition 5)**.** *If V and W are finite-dimensional real inner-product spaces, if K is a closed convex cone in V, and if $L : V \to W$ is linear and invertible, then $\beta(K) = \beta(L(K))$.*

This lightens the burden upon the reader, since all Lie groups encountered will be matrix groups equipped with the usual topology. For example, the topological group of isometries on $\mathbb{R}^n$ is the group of orthogonal matrices,

$$\mathbb{O}^n := \left\{ Q \in \mathrm{Aut}(\mathbb{R}^n) \;\middle|\; Q^T = Q^{-1} \right\},$$

with the induced metric topology. Compact means closed and bounded, and the topological interior of $X \subseteq \mathbb{R}^n$ will be indicated by $\mathrm{int}(X)$. By way of contrast with the linear dimension, we write $\mathrm{mdim}(X)$ for the manifold-dimension of a manifold X. If S denotes the unit sphere in $\mathbb{R}^3$, for example, then $\dim(\mathrm{span}(S)) = 3$ but $\mathrm{mdim}(S) = 2$. Finally, if G is a group of matrices, we write $G_x := \{g \in G \mid gx = x\}$ for the *stabilizer* or *isotropy subgroup* of G.

So far we have spoken only of closed convex cones, but one final simplifying assumption can be made. A *proper cone* is a closed convex cone K in $\mathbb{R}^n$ that also satisfies $K \cap -K = \{0\}$ and $\dim(\operatorname{span}(K)) = n$. Orlitzky [7] gives a formula for the Lyapunov rank of any closed convex cone in terms of a proper subcone. We therefore need only consider proper cones.

2 Motivation and background results

We will show that the Lorentz cone possesses the largest Lyapunov rank of any proper cone in $\mathbb{R}^n$, thereby supplying a tight upper bound on the rank of any other. Previous upper bounds were attained through "brute force" arguments [5, 8]. The conjecture that the Lorentz cone maximizes Lyapunov rank comes from two observations. The first is that the Lyapunov ranks of countless cones have been catalogued, and the rank of the Lorentz cone is simply so far supreme. The second is that, among other proper cones, the Lyapunov ranks of the *symmetric cones* have proven preternatural [5]. This leads to an informal inkling that symmetry and Lyapunov rank are intertwined.

Proper cones extend indefinitely in some direction, in the sense that they are contained in a single half-space. More specifically, every proper cone consists of nonnegative scalar multiples of a compact convex *base* that can be explicitly constructed. Suppose that K and thus K^* are proper cones in $\mathbb{R}^n$. Choose a point $e \in \operatorname{int}(K^*)$, and define the base $B \coloneqq \{x \in K \mid \langle x, e \rangle = 1\}$. Then B is compact and convex, does not contain the origin, and satisfies $K = \{\alpha b \mid \alpha \geq 0, b \in B\}$. If we pick some b_0 in the relative interior of B and translate to $B_0 \coloneqq B - b_0$, then it follows that K is the set of all nonnegative multiples of the set $b_0 + B_0$, where $b_0 \in \operatorname{int}(K)$ and B_0 is contained in a subspace of dimension $n-1$ by construction. One then thinks of b_0 as being a direction in which K extends indefinitely, with B_0 evincing the shape of the cone. The Lorentz cone thusly extends in the direction of $b_0 = e_1$, with B_0 the unit ball in $\mathbb{R}^{n-1}$. It's a challenge to conceive a compact convex set more symmetric than the unit ball. Perhaps this superlative symmetry explains its unparalleled Lyapunov rank?

Instead of studying the space of matrices Lyapunov-like on a given cone, we will attack the underlying Lie group of automorphisms. The isotropy subgroup of cone automorphisms that fix b_0 can be used to "factor out" the direction b_0, leaving us with only the shape B_0 that arises from a cross-section of the cone. Since the cross-section of the Lorentz cone is maximally symmetric, this process will show that the automorphism group of any other proper cone is in some sense inferior to that of the automorphism group of the Lorentz cone. Taking Lie algebras and tallying dimensions then proves the result.

To formalize all of this, we require a few textbook results, enumerated here without ceremony.

Proposition 2 (Faraut and Korányi [2], I.1.8)**.** *If K is a proper cone in a Euclidean space, then* $\operatorname{Aut}(\operatorname{int}(K))_x$ *is compact for any* $x \in \operatorname{int}(K)$*.*

Theorem 1 (Lee [6], 8.37)**.** *If G is a Lie group, then* $\operatorname{mdim}(G) = \dim(\operatorname{Lie}(G))$*.*

Theorem 2 (Lee [6], 20.12)**.** *If G is a Lie group and if H is a closed subgroup of G, then H is an embedded Lie subgroup of G.*

Theorem 3 (Lee [6], 21.20)**.** *If a Lie group G acts transitively on a set K and if G_x is closed for some $x \in K$, then $\operatorname{mdim}(K) = \operatorname{mdim}(G) - \operatorname{mdim}(G_x)$.*

Theorem 4 (Bröcker and tom Dieck [1], II.1.7)**.** *If G is a compact subgroup of $\operatorname{Aut}(\mathbb{R}^n)$, then there exists an inner product on $\mathbb{R}^n$ under which every element of G is an isometry.*

3 Main results

This first result is not new, only convenient.

Proposition 3. *If K is a proper cone in $\mathbb{R}^n$ and if $x \in \operatorname{int}(K)$, then the isotropy subgroup $\operatorname{Aut}(K)_x$ of the Lie group $\operatorname{Aut}(K)$ is a compact Lie group.*

Proof. That $\operatorname{Aut}(K)$ is a Lie group can be taken as given, since the Lyapunov rank is the dimension of its Lie algebra. The compactness of $\operatorname{Aut}(K)_x$ follows from Proposition 2 and the elementary fact that $\operatorname{Aut}(\operatorname{int}(K)) = \operatorname{Aut}(K)$. Since we are working with matrices, compact means closed, and closed subgroups of Lie groups are themselves Lie groups by Theorem 2. ☐

Every proper cone in $\mathbb{R}^n$ is a manifold (with boundary) of dimension n, being an intersection of closed half-spaces whose interior is nonempty. Thus their interiors are manifolds (without boundary) of dimension n as well; in particular, Proposition 3 can be combined with Theorem 3 for a symmetric cone, whose automorphism group by definition acts transitively on its interior [2].

Corollary 1. *If x belongs to the interior of some symmetric cone K in $\mathbb{R}^n$, then $\operatorname{mdim}(\operatorname{Aut}(K)) - \operatorname{mdim}(\operatorname{Aut}(K)_x) = n$.*

For the purpose of pedagogy, we will recalculate the Lyapunov rank of the Lorentz cone using Corollary 1. An application of Theorem 1 shows that

$$\dim\left(\operatorname{Lie}\left(\operatorname{Aut}\left(\mathcal{L}_+^n\right)\right)\right) = \operatorname{mdim}\left(\operatorname{Aut}\left(\mathcal{L}_+^n\right)\right) = n + \operatorname{mdim}\left(\operatorname{Aut}\left(\mathcal{L}_+^n\right)_x\right)$$

for any $x \in \operatorname{int}\left(\mathcal{L}_+^n\right)$, but we need to compute one of the isotropy subgroups $\operatorname{Aut}\left(\mathcal{L}_+^n\right)_x$ for this to be useful. The following choice of x is most convenient.

Lemma 1. *If $\mathcal{L}_+^n$ is the Lorentz cone in $\mathbb{R}^n$, then*

$$\operatorname{Aut}\left(\mathcal{L}_+^n\right)_{e_1} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix} \;\middle|\; Q \in \mathbb{O}^{n-1} \right\} = \mathbb{O}_{e_1}^n.$$

Proof. The first equality is Example 4.2 of Gowda and Sznajder [4]. The second is well-known [1, 6] and easy to verify. ☐

Theorem 5. *The Lyapunov rank of the Lorentz cone in $\mathbb{R}^n$ is $\left(n^2 - n\right)/2 + 1$.*

Proof. Example 7.27 of Lee [6] shows that

$$\operatorname{mdim}\left(\mathbb{O}^{n-1}\right) = \frac{1}{2}\left[(n-1)^2 - (n-1)\right] = \frac{1}{2}\left[n^2 - 3n + 2\right].$$

Combining Corollary 1, Lemma 1, and Theorem 1 thus gives

$$\beta\left(\mathcal{L}^n_+\right) = \operatorname{mdim}\left(\operatorname{Aut}\left(\mathcal{L}^n_+\right)\right) = n + \operatorname{mdim}\left(\mathbb{O}^{n-1}\right) = \frac{n^2 - n + 2}{2}. \qquad \square$$

Corollary 2. *If $x \in \operatorname{int}(K)$ for some proper cone K in $\mathbb{R}^n$ and if $\operatorname{Aut}(K)_x$ is a subgroup of $\mathbb{O}^n_{e_1}$, then $\beta(K) \le \left(n^2 - n\right)/2 + 1$.*

Proof. Apply Theorem 3 to any set $X \supseteq \{x\}$ on which $\operatorname{Aut}(K)$ acts transitively. The $\operatorname{Aut}(K)$-orbit of x itself shows that such a set exists. □

Theorem 6. *If K is a proper cone in $\mathbb{R}^n$, then $\beta(K) \le \beta\left(\mathcal{L}^n_+\right) = \frac{n^2-n}{2} + 1$.*

Proof. We exploit Corollary 2. The cone K is proper, so choose a $\xi \in \operatorname{int}(K)$. It follows from Proposition 3 that $\operatorname{Aut}(K)_\xi$ is compact, and from Theorem 4 that there is an inner product on $\mathbb{R}^n$ under which $\operatorname{Aut}(K)_\xi$ contains only isometries. Every inner product on $\mathbb{R}^n$ is of the form $(x, y) \mapsto \left\langle M^T M x, y\right\rangle = \langle Mx, My\rangle$ for some $M \in \operatorname{Aut}(\mathbb{R}^n)$, so let's suppose that M does the job here; that $\langle MAx, MAy\rangle = \langle Mx, My\rangle$ for all $x, y \in \mathbb{R}^n$ and all $A \in \operatorname{Aut}(K)_\xi$. There also exists some $Q \in \mathbb{O}^n$ such that $Q\left(M\xi / \|M\xi\|\right) = e_1$. Since Q is an isometry, we have $\langle QMAx, QMAy\rangle = \langle Mx, My\rangle$ for all $x, y \in \mathbb{R}^n$ and all $A \in \operatorname{Aut}(K)_\xi$. Letting $x = M^{-1}Q^{-1}w$ and $y = M^{-1}Q^{-1}z$, we observe that

$$\left\langle QMAM^{-1}Q^{-1}w, QMAM^{-1}Q^{-1}z\right\rangle = \langle w, z\rangle$$

for all $w, z \in \mathbb{R}^n$ and all $A \in \operatorname{Aut}(K)_\xi$. Thus $QM\operatorname{Aut}(K)_\xi M^{-1}Q^{-1} \subseteq \mathbb{O}^n$, since it consists of isometries on $\mathbb{R}^n$. It also fixes e_1, for if $A \in \operatorname{Aut}(K)_\xi$, then

$$QMAM^{-1}Q^{-1}e_1 = QMAM^{-1}\frac{M\xi}{\|M\xi\|} = \frac{QMA\xi}{\|M\xi\|} = \frac{QM\xi}{\|M\xi\|} = e_1.$$

As a result $QM\operatorname{Aut}(K)_\xi M^{-1}Q^{-1}$ is a subgroup of $\mathbb{O}^n_{e_1}$, namely

$$QM\operatorname{Aut}(K)_\xi M^{-1}Q^{-1} = \operatorname{Aut}(QM(K))_{e_1} \subseteq \mathbb{O}^n_{e_1}.$$

Apply Corollary 2 to conclude that the Lyapunov rank of the proper cone $QM(K)$ is at most $\left(n^2 - n\right)/2 + 1$. The result for K itself then follows from Proposition 1, and the bound is tight by virtue of Theorem 5. □

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